Distributed coordination of power generators for a linearized optimal power flow problem

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Abstract— **This paper considers the problem of optimally dispatching a set of generators in a power system; these generators are interconnected to some loads via a network of buses. We consider scenarios when the power network is operating at a steady state, and a small change in the load occurs at some of the load buses. Upon occurrence of this event, the network seeks to find the change in generator injections and voltage phase angles that makes the new steady state meet the modified load with minimum total generation cost (corresponding to the summation of the individual convex cost functions of the generating units). The resulting optimization problem is nonconvex due to the nonconvex power balance constraints at the buses. We consider a convex approximation of the problem where the power balance constraints are linearized around the initial steady-state operating point. Assuming that each bus can communicate with buses connected to it in the physical power network, we provide two provably correct continuous-time distributed strategies that allow the generators to find the optimal power set points. Both designs build on the saddle-point dynamics of the Lagrangian of the optimization problem. Various simulations illustrate our results.**

I. Introduction

In light of the increased penetration of distributed energy resources (DERs) in power systems, distributed algorithms will play a pivotal role. Distributed methods enjoy many properties, e.g., they are efficient in handling uncertain, time-varying load/generation; providing plug-and-play capabilities; guaranteeing robustness against failures; and preserving adequately the privacy of the entities involved. Motivated by this vision, in this paper we provide distributed solution strategies for optimally dispatching a set of generators in a power system. We assume that each bus is either connected to a generator or a load, and initially the network is operating at a steady state. At a certain time instant, the network experiences a small change in load at some of the buses. At that instant, buses in the network seek a new steady-state operating point that meets the new load while minimizing the total cost of generation, satisfying the physical box constraints on power generation and flow. Since this problem is nonconvex, we consider a linearized version, termed *linearized optimal power flow problem*, where the power balance constraints are linearized around the initial steady-state operating point. Our aim is then to design provably correct distributed solution strategies for the linearized problem.

Literature review: The design of distributed algorithms for optimizing power dispatch has garnered much recent attention. Some of these methods find a power allocation that meets the load and minimizes the cost but is oblivious to line flow limits, see [1], [2], and references therein for a survey of these methods. Alternatively, finding the power allocation that minimizes the cost while satisfying line flow limits is a nonconvex problem, termed as the optimal power flow (OPF) problem. Given its nonconvex nature, there are broadly three kinds of approaches to solve the OPF problem (exactly or approximately) in a distributed manner. In the first approach, the OPF problem is relaxed to yield a semidefinite program (SDP) and then, distributed strategies are designed to solve this SDP [3], [4]. These strategies involve (i) partitioning the grid into subareas; (ii) decomposing the large-scale SDP into several small-scale ones, each corresponding to a subarea; and (iii) designing iterative primal-dual coordination between these subareas. Due to these steps, this method is not easily amenable to handle networks that constantly change structure due to intermittent generation of renewables. In the second approach, the OPF problem is written in rectangular coordinates and distributed continuous-time primal-dual dynamics are employed to solve it [5]. However, this method requires quadratic cost functions and the convergence guarantee is only local. In the third approach, the OPF problem is approximated by a convex optimization problem that has affine constraints. The key step is to find a linear approximation of the nonconvex power flow constraints such that the resulting convex optimization problem is amenable to the design of distributed methods. This is the approach that we adapt in our work. While the solution of the SDP relaxation maps to the solution of the OPF problem in many cases, the linearization-based approach always yields an approximate solution. However, the structure of the linear approximation is more amenable to the design of distributed algorithms that can handle changing network structures.

The simplest of these approximations is the dc power flow model, where power losses are neglected [6]. Under this model, the work [7] designs a distributed method to find the optimizer of the relaxed convex problem. This optimizer is a good approximation of the solution of the OPF problem if the resistance of the power lines are negligible. A more general formulation is when the power losses are not neglected and instead are approximated by linearizing the power flow equations at an operating point. This is our approach in the current work. The linearization of the power balance equations used in our work is borrowed from [8], [9], [10], where such a scheme is used for estimating the distribution and shift factors in a power network. Our approach builds on the growing body of work in the area of saddle-point dynamics

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associated to the Lagrangian of the optimization problem, see e.g. [11], [12], [13], [14] and references therein.

Statement of contributions: We start with the formulation of the linearized optimal power flow (ℓOPF) problem: to determine the optimal change in generation levels and in voltage phase angles to meet the change in load encountered by the network at a steady state. The setup takes also into account generation and line flow limits. The ℓOPF problem is closely related to but more general than the dc OPF problem because ℓOPF problem also models, up to a linear approximation, the power losses in the lines. The individual costs of the generators are convex and all the constraints (equality and inequality) are affine, giving rise to a constrained convex optimization problem. Building on the saddle-point dynamics of the associated Lagrangian, we provide two distributed continuous-time dynamics that provably find the solution of the ℓOPF problem asymptotically. We present the design and analysis of the first dynamics for a general convex optimization problem and then apply it to the ℓOPF problem. We explain how both dynamics can be implemented in a distributed manner by the buses in the network via local information. A key property of these dynamics is that they both converge to the optimum even when the objective function is not strictly convex in the decision variables. Simulations illustrate our results. For reasons of space, proofs are omitted and will appear elsewhere.

II. PRELIMINARIES

This section introduces our notation and basic notions on discontinuous dynamical systems and graph theory.

1) Notation: Let $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the set of real and nonnegative real numbers, respectively. We let $\|\cdot\|$ denote the 2-norm on $\mathbb{R}^n$. Given $x, y \in \mathbb{R}^n$, x_i denotes the i-th component of x, and $x \leq y$ denotes $x_i \leq y_i$ for $i \in \{1, \ldots, n\}$. For vectors $u \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, the vector $(u; w) \in \mathbb{R}^{n+m}$ denotes their concatenation. For a matrix $A \in \mathbb{R}^{n \times m}$, we let $[A]_i$ denote its i-th row. For $a \in \mathbb{R}$, we let $[a]^+ = \max\{0, a\}$. For $a, b \in \mathbb{R}$, we let

$$[a]_b^+ = \begin{cases} a, & \text{if } b > 0, \\ \max\{0, a\}, & \text{if } b = 0. \end{cases}$$

For vectors $a, b \in \mathbb{R}^n$, $[a]_b^+$ denotes the vector the i-th component of which is $[a_i]_{b_i}^+$, $i \in \{1, \ldots, n\}$. We denote the cardinality of a set $\mathcal{S}$ by $|\mathcal{S}|$. For $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $(x, y) \mapsto F(x, y)$, we denote its partial derivative with respect to the first argument by $\nabla_x F$ and with respect to the second argument by $\nabla_y F$. The higher-order derivatives follow the convention $\nabla_{xy} F = \frac{\partial^2 F}{\partial x \partial y}$, $\nabla_{xx} F = \frac{\partial^2 F}{\partial x^2}$, and so on. A function $L : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is convex-concave on $\mathcal{X} \times \mathcal{Y}$ if, given any point $(\tilde{x}, \tilde{y}) \in \mathcal{X} \times \mathcal{Y}$, $x \mapsto L(x, \tilde{y})$ is convex and $y \mapsto L(\tilde{x}, y)$ is concave. When the context is clear, we refer to this as L being convex-concave in (x, y). A point $(x_*, y_*) \in \mathcal{X} \times \mathcal{Y}$ is a saddle point of L on the set $\mathcal{X} \times \mathcal{Y}$ if $L(x_*, y) \leq L(x_*, y_*) \leq L(x, y_*)$, for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be Lebesgue measurable and locally bounded and consider the differential equation $\dot{x} = f(x)$. A map $\gamma : [0, T) \to \mathbb{R}^n$ is a *(Caratheodory) solution* of this dynamics on the interval $[0, T)$ if it is absolutely continuous on $[0, T)$ and satisfies $\dot{\gamma}(t) = f(\gamma(t))$ almost everywhere in $[0, T)$. We use the words solution and trajectory interchangeably. For more details, we refer to [15].

Let $\mathcal{K} \subset \mathbb{R}^n$ be a closed convex set. Given a point $y \in \mathbb{R}^n$, the (point) projection of y onto $\mathcal{K}$ is $\mathrm{proj}_{\mathcal{K}}(y) = \mathrm{argmin}_{z \in \mathcal{K}} \|z - y\|$. Given $x \in \mathcal{K}$ and $v \in \mathbb{R}^n$, the (vector) projection of v at x with respect to $\mathcal{K}$ is

$$\Pi_{\mathcal{K}}(x, v) = \lim_{\delta \to 0^+} \frac{\mathrm{proj}_{\mathcal{K}}(x + \delta v) - x}{\delta}.$$

Given a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ and a closed convex set $\mathcal{K} \subset \mathbb{R}^n$, the associated projected dynamical system is

$$\dot{x} = \Pi_{\mathcal{K}}(x, f(x)), \quad x(0) \in \mathcal{K}. \tag{1}$$

For any point x in the interior of $\mathcal{K}$, we have $\Pi_{\mathcal{K}}(x, f(x)) = f(x)$. At any boundary point of $\mathcal{K}$, the projection restricts f so that the solutions of (1) remain in $\mathcal{K}$.

2) Graph theory: Following [16], a *directed graph*, or simply *digraph*, is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the vertex set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. A digraph is *undirected* if $(v, u) \in \mathcal{E}$ if and only if $(u, v) \in \mathcal{E}$. A path is an ordered sequence such that any ordered pair of vertices appearing consecutively is an edge. An undirected graph is *connected* if there is a path between any pair of distinct vertices. For a digraph, $\mathcal{N}_{v_i}^+$ and $\mathcal{N}_{v_i}^-$ are the sets of out- and in-neighbors of v_i, respectively, i.e., $\mathcal{N}_{v_i}^+ = \{v_j \in \mathcal{V} \mid (v_i, v_j) \in \mathcal{E}\}$ and $\mathcal{N}_{v_i}^- = \{v_j \in \mathcal{V} \mid (v_j, v_i) \in \mathcal{E}\}$.

III. PROBLEM STATEMENT

We consider a power network with $n \in \mathbb{Z}_{\geq 1}$ generators and $l \in \mathbb{Z}_{\geq 1}$ loads. Each bus in the network has either a generator or a load connected to it. For convenience, we index the buses as $\mathcal{V} = \{1, 2, \ldots, n, n+1, \ldots, n+l\}$, where the first n buses correspond to generators, and the next l correspond to loads. The physical interconnection between the buses is given by a connected undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where elements of $\mathcal{E}$ correspond to pairs of buses interconnected by a transmission line. We arbitrarily assign a direction to each edge in $\mathcal{E}$ and denote the resulting digraph as $\mathcal{G}_d = (\mathcal{V}, \mathcal{E}_d)$. The real power $f_{ij}^{(i)}$ entering into the line $(i, j) \in \mathcal{E}_d$ at bus i and the real power $f_{ij}^{(j)}$ flowing out of the line (i, j) at bus j are positive when the corresponding flow is in the direction assigned to the edge (i, j). Note that $f_{ij}^{(i)}$ and $f_{ij}^{(j)}$ are not necessarily equal due to power losses in the line. The power that generator $i \in \{1, \ldots, n\}$ produces and injects into the bus i is $u_i \in \mathbb{R}_{\geq 0}$ and the power drawn by the load $i \in \{1, \ldots, l\}$ at bus $n + i$ is $\ell_i \in \mathbb{R}_{\geq 0}$. The voltage magnitude and phase angle at bus $i \in \{1, \ldots, n+l\}$ are $v_i \in \mathbb{R}_{>0}$ and $\theta_i \in \mathbb{R}$, respectively. Each generator $i \in \{1, \ldots, n\}$ has a cost function $C_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, which we assume to be convex and twice continuously differentiable. The cost incurred by i in generating power u_i is $C_i(u_i)$. Each generator i has lower and upper bounds on the power it can generate, denoted by $\underline{u}_i, \overline{u}_i \in \mathbb{R}_{>0}$, respectively, with $\underline{u}_i < \overline{u}_i$. Further, the maximum allowed power flow on each line $(i, j) \in \mathcal{E}_d$, in either direction, is $\overline{f}_{ij} \in \mathbb{R}_{>0}$. The network is

at a *steady-state operating point* $(u, \ell, \theta, v) \in \mathbb{R}^n_{\geq 0} \times \mathbb{R}^l_{\geq 0} \times \mathbb{R}^{n+l} \times \mathbb{R}^{n+l}_{>0}$, if the following constraints are met:
(i) *Power balance:* for each bus $i \in \{1, \ldots, n\}$,

$$\sum_{j \in \mathcal{N}_i^+} f_{ij}^{(i)} - \sum_{j \in \mathcal{N}_i^-} f_{ji}^{(i)} = u_i, \tag{2}$$

and for each bus $i \in \{n+1, \ldots, n+l\}$,

$$\sum_{j \in \mathcal{N}_i^+} f_{ij}^{(i)} - \sum_{j \in \mathcal{N}_i^-} f_{ji}^{(i)} = -\ell_{i-n}, \tag{3}$$

where $\mathcal{N}_i^+$ and $\mathcal{N}_i^-$ are out- and in-neighbors of bus i in the digraph $\mathcal{G}_d$ and flows on line $(i,j) \in \mathcal{E}_d$ in terms of voltage magnitudes and phase angles are given by [17],

$$f_{ij}^{(i)} = g_{ij} v_i^2 - v_i v_j (g_{ij} \cos(\theta_{ij}) + b_{ij} \sin(\theta_{ij})), \tag{4a}$$

$$f_{ij}^{(j)} = -g_{ij} v_j^2 + v_i v_j (g_{ij} \cos(\theta_{ij}) - b_{ij} \sin(\theta_{ij})), \tag{4b}$$

where g_{ij} and b_{ij} is the conductance and susceptance (i,j); (v_i, θ_i) and (v_j, θ_j) are voltage magnitude and phase angle for bus i and j; and $\theta_{ij} = \theta_i - \theta_j$.
(ii) *Box constraints:* for each generator $i \in \{1, \ldots, n\}$, $\underline{u}_i \leq u_i \leq \overline{u}_i$ and, for each power line $(i,j) \in \mathcal{E}_d$, $-\overline{f}_{ij} \leq f_{ij}^{(i)} \leq \overline{f}_{ij}$, and $-\overline{f}_{ij} \leq f_{ij}^{(j)} \leq \overline{f}_{ij}$.

Consider the situation where the network is at a steady-state operating point (u, ℓ, θ, v), and the load changes by a small quantity $\Delta\ell$. With the voltage magnitudes held constant, the network would seek a new steady-state operating point, $(\tilde{u}, \ell + \Delta\ell, \tilde{\theta}, v)$, that minimizes the total cost of generation $\sum_{i=1}^n C_i(\tilde{u}_i)$. However, this optimization problem is nonconvex because of the power flow constraints. Instead, we solve a linearized version of the problem, termed *linearized optimal power flow* (ℓOPF), which seeks to optimize the change in the generator injections Δu and phase angles $\Delta\theta$ with respect to the current steady state. This approach is justified when $\Delta\ell$ is small enough as the linear approximation provides a near optimal solution for the nonconvex problem. Moreover, any violation of the constraints of the nonconvex problem by the optimizer of the linearized problem is handled by the primary and secondary control schemes, which ensure satisfaction of the constraints at a faster time scale as compared to the scale at which dispatch decisions are made. To linearize the power flow constraints (4) around (u, ℓ, θ, v), we assume that the voltage magnitudes are kept constant. This assumption is valid for transmission networks, where changes in voltage phase angles (resp. in voltage magnitudes) are strongly coupled with changes in active power injections (resp. in reactive power injections) and weakly coupled with changes in reactive power injections (resp. in active power injections), see e.g., [17]. For distribution networks, this is generally not the case, unless one utilizes reactive-power capable DERs to provide voltage control at a much faster time scale than that associated with the changes in active-power injection, see e.g., [18]. Formally, if $\{f_{ij}^{(i)}, f_{ij}^{(j)}\}_{(i,j) \in \mathcal{E}_d}$ denote the flows at the steady state (u, ℓ, θ, v), then the ℓOPF problem is defined as

$$\underset{\Delta u, \Delta\theta, \Delta f}{\text{minimize}} \quad \sum_{i=1}^n C_i(u_i + \Delta u_i), \tag{5a}$$

subject to For all $i \in \{1, \ldots, n\}$

$$\sum_{j \in \mathcal{N}_i^+} \Delta f_{ij}^{(i)} - \sum_{j \in \mathcal{N}_i^-} \Delta f_{ji}^{(i)} = \Delta u_i, \tag{5b}$$

$$\underline{u}_i \leq u_i + \Delta u_i \leq \overline{u}_i, \tag{5c}$$

For all $i \in \{n+1, \ldots, n+l\}$

$$\sum_{j \in \mathcal{N}_i^+} \Delta f_{ij}^{(i)} - \sum_{j \in \mathcal{N}_i^-} \Delta f_{ji}^{(i)} = -\Delta\ell_{i-n}, \tag{5d}$$

For all $(i,j) \in \mathcal{E}_d$

$$-\overline{f}_{ij} \leq f_{ij}^{(i)} + \Delta f_{ij}^{(i)} \leq \overline{f}_{ij}, \tag{5e}$$

$$-\overline{f}_{ij} \leq f_{ij}^{(j)} + \Delta f_{ji}^{(i)} \leq \overline{f}_{ij}, \tag{5f}$$

$$\Delta f_{ij}^{(i)} = \alpha_{ij}(\Delta\theta_i - \Delta\theta_j), \tag{5g}$$

$$\Delta f_{ij}^{(j)} = \beta_{ij}(\Delta\theta_i - \Delta\theta_j), \tag{5h}$$

where for each $(i,j) \in \mathcal{E}_d$, we have $\alpha_{ij} = g_{ij} v_i v_j \sin(\theta_{ij}) - b_{ij} v_i v_j \cos(\theta_{ij})$, and $\beta_{ij} = -g_{ij} v_i v_j \sin(\theta_{ij}) - b_{ij} v_i v_j \cos(\theta_{ij})$. Constraints (5b) and (5d) represent the power balance at generator and load buses, respectively. Constraints (5c), (5e), and (5f) represent the min- and max- constraints on the change in generation levels and the change in power flow in the lines. Constraints (5g) and (5h) represent the linearized relationship between the change in flows and the change in voltage phase angles. Note that one can write the above optimization in terms of $(\Delta u, \Delta\theta)$ by substituting incremental flows using (5g) and (5h). We avoid doing this substitution as the current formulation helps us later in our distributed algorithm design. Denoting $c(\Delta u) = \sum_{i=1}^n c_i(\Delta u_i) = \sum_{i=1}^n C(u_i + \Delta u_i)$, one can rewrite the ℓOPF problem in a compact form as

$$\underset{\Delta u, \Delta\theta, \Delta f}{\text{minimize}} \quad c(\Delta u), \tag{6a}$$

$$\text{subject to} \quad A_1 \Delta x \leq b_1, \tag{6b}$$

$$A_2 \Delta x = b_2, \tag{6c}$$

where $A_1 \in \mathbb{R}^{2n+4|\mathcal{E}_d| \times 2n+l+2|\mathcal{E}_d|}$, $A_2 \in \mathbb{R}^{n+l+2|\mathcal{E}_d| \times 2n+l+2|\mathcal{E}_d|}$, $b_1 \in \mathbb{R}^{2n+4|\mathcal{E}_d|}$, $b_2 \in \mathbb{R}^{n+l+2|\mathcal{E}_d|}$, and as a shorthand notation, we use $\Delta x = (\Delta u; \Delta\theta; \Delta f)$ throughout this paper. In (6b), we have stacked the inequalities (5c), (5e), and (5f), in this order and in (6c), we have stacked the equalities (5b), (5d), (5g), and (5h), in this order. We assume that the optimization (6) admits a feasible solution and has finite primal and dual optimal values. Since the constraints are affine, the optimization problem satisfies the refined Slater condition and hence, the duality gap is zero [19]. We denote the set of feasible solutions of (6) by $\mathcal{F}_p$ and the set of primal-dual solutions of (6) by $\mathcal{F}_p^* \times \mathcal{F}_d^*$ where $\mathcal{F}_p^*$ (resp. $\mathcal{F}_d^*$) denotes primal (resp. dual) solutions. Note that optimization (5) is more general than the dc power flow problem [6] as we do not neglect power losses, but rather approximate them using linearization.

Our objective is to design distributed algorithms that solve the ℓOPF problem (6). By distributed we mean that each bus can determine the decision variables pertaining to its bus at the optimum, without having a centralized decision making unit solving the ℓOPF problem. To this end, we assume that the buses are decision makers and can exchange information with the buses adjacent to them in the physical network.

Remark 3.1: (Information and decision variables): Given the network's steady state (u, ℓ, θ, v), we assume each bus $i \in \{1, \dots, n+l\}$ knows the voltage magnitude and phase angle corresponding to it and its neighboring buses in $\mathcal{G}$. Bus i also knows the conductance and reactance of the power lines connecting i to its neighboring buses. Further, bus $i \in \{1, \dots, n\}$ knows u_i and bus $i \in \{n+1, \dots, n+l\}$ knows ℓ_{i-n}. With this information, each bus i computes α_{ij}, β_{ij} for all $(i,j) \in \mathcal{E}$. The decision variables for each generator bus $i \in \{1, \dots, n\}$ are $(\Delta u_i, \Delta\theta_i, \{\Delta f_{ij}^{(i)}\}_{j \in \mathcal{N}_i^{\text{out}}}, \{\Delta f_{ji}^{(i)}\}_{j \in \mathcal{N}_i^{\text{in}}})$ and for each load bus $i \in \{n+1, \dots, n+l\}$ are $((\Delta\theta)_i, \{\Delta f_{ij}^{(i)}\}_{j \in \mathcal{N}_i^{\text{out}}}, \{\Delta f_{ji}^{(i)}\}_{j \in \mathcal{N}_i^{\text{in}}})$. •

IV. DISTRIBUTED ALGORITHMIC SOLUTIONS

In this section, we present two distributed solution strategies for the ℓOPF problem. Both strategies build on the continuous-time saddle-point information of the Lagrangian of the optimization problem. As we illustrate below, they have an inherent quality of being amenable to distributed implementation. The Lagrangian of (6) is given by

$$L(\Delta x, \lambda, \mu) = c(\Delta u) + \lambda^\top (A_1 \Delta x - b_1) + \mu^\top (A_2 \Delta x - b_2), \quad (7)$$

where $\lambda \in \mathbb{R}_{\geq 0}^{2n+4|\mathcal{E}_d|}$ and $\mu \in \mathbb{R}^{n+l+2|\mathcal{E}_d|}$ are the Lagrange multipliers associated to constraints (6b) and (6c), respectively. Since the duality gap is zero, a primal-dual solution $(\Delta u^*, \Delta\theta^*, \Delta f^*, \lambda^*, \mu^*)$ of (6) is also a saddle point of L on the set $(\mathbb{R}^n \times \mathbb{R}^{n+l} \times \mathbb{R}^{2|\mathcal{E}_d|}) \times (\mathbb{R}_{\geq 0}^{2n+4|\mathcal{E}_d|} \times \mathbb{R}^{n+l+2|\mathcal{E}_d|})$ [19, Section 5.4.2]. One way of designing a convergent strategy is to employ a projected version of the saddle-point dynamics for the Lagrangian, see e.g., [20], [11], [21]. However, such strategy requires the objective function to be strictly convex in the primal variables for convergence, which need not be the case for (6). This motivates our forthcoming discussion.

A. Saddle-point dynamics for augmented Lagrangian

We provide in the Appendix the details of the design and analysis of this dynamics for a general (not necessarily strictly) convex optimization. Here, we apply that treatment for the ℓOPF problem. Let the *augmented Lagrangian* $L_{\text{aug}} : \mathbb{R}^n \times \mathbb{R}^{n+l} \times \mathbb{R}^{2|\mathcal{E}_d|} \times \mathbb{R}_{\geq 0}^{2n+4|\mathcal{E}_d|} \times \mathbb{R}^{n+l+2|\mathcal{E}_d|} \to \mathbb{R}$ be

$$L_{\text{aug}}(\Delta x, \lambda, \mu) = L(\Delta x, \lambda, \mu) + \|\phi([A_1 \Delta x - b_1]^+)\|^2 + \|A_2 \Delta x - b_2\|^2,$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is $t \mapsto \phi(t) = e^t - 1$, and with a slight abuse of notation, we use $\phi(v) = (\phi(v_1), \phi(v_2), \dots, \phi(v_m))$ for $v \in \mathbb{R}^m$. The augmented Lagrangian is convex-concave in $(\Delta x, (\lambda, \mu))$ and continuously differentiable. The saddle-point dynamics for augmented Lagrangian is

$$\frac{d\Delta x}{dt} = -\nabla_{\Delta x} L_{\text{aug}}(\Delta x, \lambda, \mu), \quad (8a)$$

$$\frac{d\lambda}{dt} = [\nabla_\lambda L_{\text{aug}}(\Delta x, \lambda, \mu)]_\lambda^+ = [A_1 \Delta x - b_1]_\lambda^+, \quad (8b)$$

$$\frac{d\mu}{dt} = \nabla_\mu L_{\text{aug}}(\Delta x, \lambda, \mu) = A_2 \Delta x - b_2. \quad (8c)$$

The following result states the convergence properties of the above dynamics. The result follows from Theorem A.2.

Theorem 4.1: (Asymptotic convergence of saddle-point dynamics for augmented Lagrangian to $\mathcal{F}_p^* \times \mathcal{F}_d^*$): Suppose there exists a primal-dual solution $(\Delta x_*, \lambda_*, \mu_*)$ of (6) that satisfies the strict complementary slackness condition, that is, for all $i \in \{1, \dots, 2n + 4|\mathcal{E}_d|\}$, either $(\lambda_*)_i \neq 0$ or $[A_1]_i \Delta x_* - (b_1)_i \neq 0$. Then, any solution of (8) starting in $\mathbb{R}^n \times \mathbb{R}^{n+l} \times \mathbb{R}^{2|\mathcal{E}_d|} \times \mathbb{R}_{\geq 0}^{2n+4|\mathcal{E}_d|} \times \mathbb{R}^{n+l+2|\mathcal{E}_d|}$ converges asymptotically to a point in $\mathcal{F}_p^* \times \mathcal{F}_d^*$.

According to Theorem 4.1, the saddle-point dynamics for augmented Lagrangian can be used to find the solution of the ℓOPF problem (5). Note that the strict complementary slackness condition is not restrictive because it is generic. That is, if there does not exist a primal-dual optimizer that satisfy this condition, a slight perturbation of the constraints gives a problem that possesses such a solution.

Remark 4.2: (Distributed implementation of saddle-point dynamics for augmented Lagrangian): Under the saddle-point dynamics for augmented Lagrangian, the primal variables of bus i correspond to its own decision variables, as discussed in Remark 3.1. Regarding the dual variables, we create a partition in the following way. Each bus $i \in \{1, \dots, n+l\}$ executes the dynamics for dual variables $(\mu_i, \{\mu_{ij}, \lambda_{ij}^{(m)}, \lambda_{ij}^{(M)}\}_{j \in \mathcal{N}_i^+}, \{\mu_{ji}^{(m)}, \lambda_{ji}^{(m)}, \lambda_{ji}^{(M)}\}_{j \in \mathcal{N}_i^-})$, where μ_i corresponds to power balance constraint at i; μ_{ij} for $j \in \mathcal{N}_i^+$ and μ_{ji} for $j \in \mathcal{N}_i^-$ corresponds to the equality constraint representing linearized relationship with phase angles for $\Delta f_{ij}^{(i)}$, $j \in \mathcal{N}_i^+$ and $\Delta f_{ji}^{(i)}$ $j \in \mathcal{N}_i^-$, respectively; $(\lambda_{ij}^{(m)}, \lambda_{ij}^{(M)})$, $j \in \mathcal{N}_i^+$ correspond to the min- and max- flow constraint on the variable $\Delta f_{ij}^{(i)}$ $j \in \mathcal{N}_i^+$; similarly, $(\lambda_{ji}^{(m)}, \lambda_{ji}^{(M)})$, $j \in \mathcal{N}_i^-$ correspond to the min- and max- flow constraint on the variable $\Delta f_{ji}^{(i)}$ $j \in \mathcal{N}_i^-$. Along with these variables, each generator bus $i \in \{1, \dots, n\}$ executes the dynamics for dual variables $(\lambda_i^{(m)}, \lambda_i^{(M)})$ corresponding to the min-, max-constraint on generation of i. Given this partition, one can write the dynamics of variables that each bus computes in terms of the information available to that bus, that is, its own state and the state of its neighbors in $\mathcal{G}$, see Remark 3.1. •

B. Saddle-point dynamics for modified Lagrangian

The solution presented here follows the exposition in [14]. First, define the *modified Lagrangian* $L_{\text{mod}} : \mathbb{R}^n \times \mathbb{R}^{n+l} \times \mathbb{R}^{2|\mathcal{E}_d|} \times \mathbb{R}^{n+l+2|\mathcal{E}_d|} \to \mathbb{R}$ as

$$L_{\text{mod}}(\Delta x, \mu) = c(\Delta u) + \frac{1}{2}\|A_2 \Delta x - b_2\|^2 + \mu^\top (A_2 \Delta x - b_2).$$

Note that L_{mod} is twice continuously differentiable and convex-concave on $(\mathbb{R}^n \times \mathbb{R}^{n+l} \times \mathbb{R}^{2|\mathcal{E}_d|}) \times (\mathbb{R}^{n+l+2|\mathcal{E}_d|})$. Let $\mathcal{F}_{pp} = \{\Delta x \in \mathbb{R}^n \times \mathbb{R}^{n+l} \times \mathbb{R}^{2|\mathcal{E}_d|} \mid \Delta x \text{ satisfy (6b)}\}$, that is, the set of points that satisfy the box constraints in (5). Define the saddle-point dynamics for modified Lagrangian as

$$\frac{d\Delta x}{dt} = \Pi_{\mathcal{F}_{pp}}\Big(\Delta x, -\nabla_{\Delta x} L_{\texttt{mod}}(\Delta x, \mu)\Big), \quad (9a)$$

$$\frac{d\mu}{dt} = \nabla_{\mu} L_{\texttt{mod}}(\Delta x, \mu) = A_2 \Delta x - b_2. \quad (9b)$$

Recall that the vector field given $(\Delta x, \xi) \mapsto \Pi_{\mathcal{F}_{pp}}(\Delta x, \xi)$ restricts the flow $\xi \in \mathbb{R}^n \times \mathbb{R}^{n+l} \times \mathbb{R}^{2|\mathcal{E}_d|}$ to the set $\mathcal{F}_{pp}$, cf. Section II. The next result states the convergence properties of (9) and is a direct consequence of [14, Theorem 5.6].

Theorem 4.3: (Asymptotic convergence of saddle-point dynamics for modified Lagrangian to $\mathcal{F}_p^*$): Any trajectory of (9) starting in $\mathcal{F}_{pp} \times \mathbb{R}^{n+l+2|\mathcal{E}_d|}$ is bounded and it converges asymptotically to a point in $\mathcal{F}_p^* \times \mathbb{R}^{n+l+2|\mathcal{E}_d|}$.

The above result implies that the saddle-point dynamics for modified Lagrangian can be used as an algorithmic solution for the ℓOPF problem (6). Also, since each term of $L_{\texttt{mod}}$ is also a term of $L_{\texttt{aug}}$, the distributed implementation of (9) follows from Remark 4.2 using the fact that for each bus, the projection in (9a) for variables of a bus can be computed by the bus using the information available to it.

V. Simulations

Here, we illustrate the application of saddle-point dynamics for augmented Lagrangian (8) and saddle-point dynamics for modified Lagrangian (9) to solve the ℓOPF problem (5) for a 9-bus network with 3 generator buses and 9 power lines. This example is a modification of the 9 bus case study from MATPOWER [22], and for space reasons we only document here the changes made to this example. We assume that (1) shunt conductances and reactances at each bus are zero; (2) line charging susceptance for each line is zero; (3) the cost function for generator i is $C_i(u_i) = a_i u_i^2 + b_i u_i$ [$\$/(100\text{MW})$], with coefficients for the three generators given by $a = (0.11; 0.085; 0.1225)$ and $b = (5; 1.2; 1)$. Figure 1 shows the evolution of the optimization variables $(\Delta u, \Delta\theta, \Delta f)$ and the total cost $c(\Delta u)$ for the dynamics (8) and (9). As established in Theorem 4.1 and 4.3, the trajectories converge to an optimizer. Moreover, in this particular example, the converged solution is the same.

VI. Conclusions

We have formulated the ℓOPF problem for a power network consisting of generator and load buses by linearizing the nonconvex power balance constraints around an operating point of the network. We have provided two continuous-time distributed dynamics that allow the buses to asymptotically find the optimizer of the ℓOPF problem. The design and analysis of the first of these dynamics, termed saddle-point dynamics for augmented Lagrangian, is novel and studied for a general constrained convex optimization problem over a network. Future work will explore the robustness of the dynamics against asynchronicity in the network, communication link failures, and noisy updates. We also intend to study the interplay of these dynamics with the network frequency dynamics and the different layers of generator controllers.

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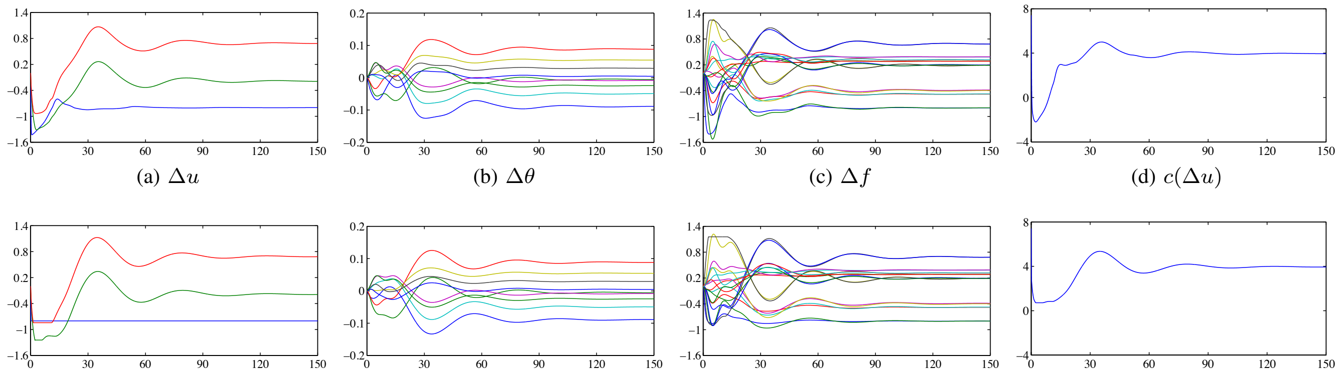

(a) Δu (b) $\Delta\theta$ (c) Δf (d) $c(\Delta u)$

(e) Δu (f) $\Delta\theta$ (g) Δf (h) $c(\Delta u)$

Fig. 1. Illustration of the execution of saddle-point dynamics for augmented Lagrangian (plots (a)-(d)) and for modified Lagrangian (plots (e)-(h)) for a 9 bus example from MATPOWER described in Section V. We assume that before $t = 0$, the network is at a steady-state operating point described by (1) the real power injection at the generator buses (numbered $1-3$) $u = (90.1, 134.44, 94.31)$ MW; (2) the real power load at the load buses (numbered $4-9$) $\ell = (0, 90, 0, 100, 0, 125)$ MW; (3) the voltage magnitude at buses $v = (1.1,\ 1.1,\ 1.1,\ 1.0648,\ 1.0406,\ 1.08,\ 1.0575,\ 1.0744,\ 1.0276)$ p.u., with base voltage 345 kV at each bus; and (4) the voltage phase angle at buses $\theta = (0, 0.0874,\ 0.0563,\ -0.0443,\ -0.0702,\ 0.0098,\ -0.0218,\ 0.0163,\ -0.0832)$ radians. At time, $t = 0$, the load at each bus decreases by 10%. This defines the ℓOPF problem (5) that we wish to solve. The initial condition for both set of plots and all variables is zero. Both dynamics converge to values $\Delta u^* = (-0.8, -0.193, 0.681)$ MW, $\Delta\theta^* = (-0.0886,\ -0.0057,\ 0.0881,\ -0.0493,\ -0.0082,\ 0.0545,\ 0.0292,\ 0.0045,\ -0.0245)$ radians and $\Delta f^* = (-0.8011,\ -0.7999,\ -0.4822,\ -0.4766,\ -0.3861,\ -0.3997,\ 0.6818,\ 0.6824,\ 0.283,\ 0.2817,\ 0.3821,\ 0.3863,\ 0.1921,\ 0.1926,\ 0.1945,\ 0.1878,\ 0.3133,\ 0.3172)$ MW. The total generation cost converges to 0.3175.

Appendix

Consider the convex optimization problem on $\mathbb{R}^n$

$$\begin{aligned} &\text{minimize} && f(x), && \text{(A.10a)}\\ &\text{subject to} && g(x) \leq 0, && \text{(A.10b)}\\ & && Ax = b, && \text{(A.10c)}\end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable convex functions, $A \in \mathbb{R}^{p\times n}$, and $b \in \mathbb{R}^p$. We specifically consider the case when f is not necessarily strictly convex, which is a common requirement to guarantee the asymptotic convergence of the saddle-point dynamics to the primal-dual optimizers, cf. [20], [11], [21]. We address this obstacle by modifying the inequality constraints following an idea proposed in [11, Section 4.1]. Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex with $\phi(0) = 0$, $\nabla\phi(t) > 0$ and $\nabla^2\phi(t) > 0$ for all $t \in \mathbb{R}$. Modify (A.10b) using ϕ and write

$$\begin{aligned} &\text{minimize} && f(x), && \text{(A.11a)}\\ &\text{subject to} && \phi(g(x)) \leq 0, && \text{(A.11b)}\\ & && Ax = b, && \text{(A.11c)}\end{aligned}$$

where for convenience we use $\phi(g(x)) = (\phi(g_1(x)), \phi(g_2(x)), \dots, \phi(g_m(x)))$. Note that optimizers of (A.10) and (A.11) are same. Assume that Slater's condition is satisfied for (A.10). This implies that the same is true for (A.11) and hence, the duality gap is zero [19]. The Lagrangian of (A.11) is $L : \mathbb{R}^n \times \mathbb{R}^m_{\geq 0} \times \mathbb{R}^p \to \mathbb{R}$,

$$L(x, \lambda, \mu) = f(x) + \textstyle\sum_{i=1}^m \lambda_i \phi(g_i(x)) + \mu^\top (Ax - b),$$

where $\lambda \in \mathbb{R}^m_{\geq 0}$ and $\mu \in \mathbb{R}^p$ are the Lagrange multipliers associated with (A.11b) and (A.11c), respectively. The Lagrangian is convex-concave in $(x, (\lambda, \mu))$ [19]. A point (x_*, λ_*, μ_*) is a primal-dual solution of (A.11) if and only if it satisfies the Karush-Kuhn-Tucker (KKT) conditions

$$\nabla_x L(x_*, \lambda_*, \mu_*) = 0, \quad \textstyle\sum_{i=1}^m (\lambda_*)_i \phi(g_i(x_*)) = 0,$$

$$\phi(g(x_*)) \leq 0, \quad Ax_* = b, \quad \lambda_* \geq 0.$$

Let the augmented Lagrangian $L_{\mathtt{aug}}$ be defined as

$$L_{\mathtt{aug}}(x, \lambda, \mu) = L(x, \lambda, \mu) + \|Ax - b\|^2 + \|[\phi(g(x))]^+\|^2.$$

Note that $L_{\mathtt{aug}}$ is convex-concave in $(x, (\lambda, \mu))$ as L is convex-concave in $(x, (\lambda, \mu))$ and the map $x \mapsto \|Ax-b\|^2 + \|[\phi(g(x))]^+\|^2$ is convex. Also, $L_{\mathtt{aug}}$ is continuously differentiable. Next result states that the primal-dual solutions of (A.11) are the saddle points of the augmented Lagrangian.

Lemma A.1: (Primal-dual solutions of (A.11) *are saddle points of L_{aug}):* A point $(x_*, \lambda_*, \mu_*) \in \mathbb{R}^n \times \mathbb{R}^m_{\geq 0} \times \mathbb{R}^p$ is a primal-dual optimizer of (A.11) if and only if it is a saddle point of $L_{\mathtt{aug}}$ on the set $(\mathbb{R}^n) \times (\mathbb{R}^m_{\geq 0} \times \mathbb{R}^p)$.

As a consequence of Lemma A.1, our strategy to solve (A.11) is to find the saddle points of the augmented Lagrangian on $(\mathbb{R}^n) \times (\mathbb{R}^m_{\geq 0} \times \mathbb{R}^p)$. To this end, we define the (projected) saddle-point dynamics for $L_{\mathtt{aug}}$ as

$$\begin{aligned} \dot{x} &= -\nabla_x L_{\mathtt{aug}}(x, \lambda, \mu) = -\nabla f(x) - 2A^\top (Ax - b) - A^\top \mu \\ &\quad - \nabla \|[\phi(g(x))]^+\|^2 - \sum_{i=1}^m \lambda_i \nabla\phi(g_i(x)) \nabla g_i(x), \quad \text{(A.13a)}\\ \dot{\lambda} &= [\nabla_\lambda L_{\mathtt{aug}}(x, \lambda, \mu)]^+_\lambda = [\phi(g(x))]^+_\lambda, \quad \text{(A.13b)}\\ \dot{\mu} &= \nabla_\mu L_{\mathtt{aug}}(x, \lambda, \mu) = Ax - b. \quad \text{(A.13c)}\end{aligned}$$

Note that a point in $\mathbb{R}^n \times \mathbb{R}^m_{\geq 0} \times \mathbb{R}^p$ is an equilibrium of the above dynamics if and only if it is a primal-dual optimizer of (A.11). Next, we state the convergence result.

Theorem A.2: (Asymptotic convergence of (A.13)*):* Assume there exists a primal-dual optimizer (x_*, λ_*, μ_*) of (A.11) satisfying the strict complementary slackness condition, that is, for all $i \in \{1, \dots, m\}$, either $(\lambda_*)_i \neq 0$ or $g_i(x_*) \neq 0$. Then, any trajectory of (A.13) starting in $\mathbb{R}^n \times \mathbb{R}^m_{\geq 0} \times \mathbb{R}^p$ converges asymptotically to a point in the set of primal-dual optimizers of (A.11).