Exploiting Phase Cohesiveness for Frequency Control of Islanded Inverter-Based Microgrids

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***Abstract*— In this paper, we propose a generation control strategy for islanded lossless microgrids with inverter-interfaced generators. In particular, we address the problem of obtaining a desired active power load sharing while regulating the frequency to some nominal value. The proposed control strategy is based on the slow readjustment of the active power reference of each inverter, i.e., each inverter aims to track a slowly-varying reference that is set sufficiently close to the actual active power injection, and, thus easy to track. The control enforces the trajectory to satisfy the so-called phase-cohesiveness property, i.e., the absolute value of the voltage phase angle difference across electrical lines is smaller than $\frac{\pi}{2}$, which ensures the system remains stable at all times, while achieving the desired active power load sharing. We also propose a method to find the active power reference distributively which satisfies the phase-cohesiveness property for tree networks as well as for some cyclic networks.**

I. Introduction

Generally, a collection of interconnected electrical loads and generators is considered a microgrid if, compared with the bulk power system, it has a substantially smaller physical footprint, the generators have lower ratings, and it is able to operate in an islanded mode (see, e.g., [1], [2]). The use of microgrids has the potential to be an effective solution for efficiently managing the increased penetration of Distributed Energy Resources (DERs); however, there are several control problems that must be addressed to ensure reliable and economical operation of microgrids. In this paper, we address one such problem; namely, the microgrid generation control problem, with special emphasis on achieving active power load sharing and frequency regulation.

The generation control problem in microgrids has been extensively studied by other researchers. For example, in [3], the authors propose a distributed controller that synchronizes inverters, drives the frequency to the nominal value, and achieves active power load sharing under small load perturbations. In [4], the authors propose a distributed secondary controller, the implementation of which is based on a linearized microgrid model and a consensus-type distributed algorithm, to eliminate frequency deviation that results from the operation of the primary droop controller. In [5], the authors present a distributed secondary controller which computes the average of inverter frequencies; then, each inverter uses a PI controller to drive this average frequency to the desired value. In [2], the authors present a hierarchical generation control architecture, and implement secondary and tertiary control using a distributed approach based on the so-called ratio-consensus algorithm [6].

Although a substantial effort has been put into addressing the generation control problem in microgrids, many issues, including stability, synchronization and proper active power load sharing, have not been completely resolved. Most control designs in the literature are based on linearized models, and can guarantee stability and good performance only locally under small perturbations in the load (see, e.g., [3], [4]). Moreover, because islanded microgrids often lack enough rotating inertia to accommodate large load perturbations (which occur because of a low load factor), large frequency deviations might result, leading to instability (see, e.g., [7], [8]). Several works approach this problem by incorporating additional energy reserves to increase the rotating inertia of the system (see, e.g., [8], [9]).

Our work in this paper aims to achieve frequency regulation and appropriate active power load sharing among inverter-interfaced generators regardless of the size of the load perturbations by designing appropriate controls. To this end, we adopt the lossless Kuramoto-type inverter-based microgrid model proposed in [10], and design a control that enforces the absolute value of the voltage phase angle difference across electrical lines to be strictly smaller than $\frac{\pi}{2}$—a property referred to as phase-cohesiveness (see, e.g., [10], [11]). This control guarantees global convergence to the desired active power reference, and regulation of the frequency to some nominal value. The reference is computed distributively by solving a convex optimization problem that enforces the phase-cohesiveness property. We illustrate the operation of the proposed control via numerical simulations. All results in lemmas and propositions are stated without proofs.

II. Preliminaries

In this section, we present the droop-controlled inverter-based microgrid model adopted in this work, and formulate the control objectives to be met.

A. Microgrid Dynamic Model

Consider a collection of droop-controlled inverters and loads interconnected by a lossless network. The topology of this network can be described by an undirected graph, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} = \mathcal{V}^{(I)} \cup \mathcal{V}^{(L)}$, where $\mathcal{V}^{(I)} = \{1, \dots, m\}$ denotes the set of nodes with an inverter, and $\mathcal{V}^{(L)} = \{m+1, \dots, n\}$ denotes the set of nodes with a load; and where $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, with $\{i, j\} \in \mathcal{E}$ if nodes i and j are electrically connected.

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Let $\theta_i(t)$ and $V_i(t)$ denote, respectively, the phase angle and magnitude of the voltage at node $i \in \mathcal{V}$, and define $\theta(t) = [\theta_1(t), \ldots, \theta_n(t)]^{\mathrm{T}}$ and $V(t) = [V_1(t), \ldots, V_n(t)]^{\mathrm{T}}$. Then, the active power injected into the network via node $i \in \mathcal{V}$ is given by

$$P_i(\theta(t), V(t)) = \sum_{j \in \mathcal{V}} V_i(t) V_j(t) B_{ij} \sin(\theta_i(t) - \theta_j(t)), \quad (1)$$

with $B_{ij} = -b_{ij}$, where $b_{ij} < 0$ is the susceptance of the line connecting nodes i and j. Define a one-to-one map $\mathbb{I} : \mathcal{E} \to \mathbb{R}$ such that every e in the set $\{1, 2, \ldots, |\mathcal{E}|\}$ is arbitrarily assigned to exactly one edge $\{i, j\} \in \mathcal{E}$, i.e., $\mathbb{I}(\{i, j\}) = e$. Then, if we assign directions to each edge $\{i, j\}$ arbitrarily, e.g., i is taken to be the head and j the tail, we can define a node-to-edge incidence matrix, M, as follows: for each $e = \mathbb{I}(\{i, j\})$, $M_{ie} = 1$, $M_{je} = -1$, and zero otherwise. Then, (1) can also be written in matrix form as follows:

$$P(\theta(t), V(t)) = M\Gamma(V(t)) f(\theta(t)), \quad (2)$$

where $P(\theta(t), V(t)) = [P_1(\theta(t), V(t)), \ldots, P_n(\theta(t), V(t))]^{\mathrm{T}}$, $\Gamma(V(t))$ is a diagonal matrix with entries $\Gamma_{ee}(V(t)) = V_i(t) V_j(t) B_{ij}$, $e = \mathbb{I}(\{i, j\})$, $\{i, j\} \in \mathcal{E}$, and $f(\theta(t)) = [f_1(\theta(t)), \ldots, f_{|\mathcal{E}|}(\theta(t))]^{\mathrm{T}}$ with $f_e(\theta(t)) = \sin(\theta_i(t) - \theta_j(t))$.

Let $\omega_i(t)$ denote the frequency at node $i \in \mathcal{V}$ at time t; then, assuming the inverters implement a frequency-droop controller (see, e.g., [3]), we have that, for each $i \in \mathcal{V}^{(I)}$,

$$\omega_i(t) = \omega^* + D_i^{-1}(P_i^*(t) - P_i(\theta(t), V(t))), \quad (3)$$

where ω^* is some nominal frequency, $D_i > 0$ is the droop coefficient, $P_i^*(t)$ is the active power reference of inverter $i \in \mathcal{V}^{(I)}$; and, for each $j \in \mathcal{V}^{(L)}$,

$$0 = -\ell_j(t) - P_j(\theta(t), V(t)), \quad (4)$$

where $\ell_j(t) > 0$ is the active power demand at load j; $\ell_j(\cdot)$ is assumed to be a piecewise constant function.

In the remainder, we assume that voltage magnitudes are fixed for all nodes, i.e., $V_i(t) = V_i$ for each $i \in \mathcal{V}$ and all t; thus, for brevity, we drop the V-argument from all functions which depend on it. We also assume that after a change in load, the controller is fast enough to restore the system frequency to its nominal value before another load change occurs; thus, we assume $\ell_i(t)$ and $P_i^*(t)$ to be positive constants, which we denote by ℓ_i and P_i^*, respectively.

We transform all inverter and load node voltage angles into a rotating coordinate frame with frequency ω^* by defining $\dot{\theta}_i(t) = \omega_i(t) - \omega^*$ for $i \in \mathcal{V}$. Then, by defining $u_i(t) := \frac{P_i^*(t) - P_i(\theta(t))}{D_i}$ for $i \in \mathcal{V}^{(I)}$, we can rewrite (3) as follows:

$$\dot{\theta}_i(t) = u_i(t), \quad i \in \mathcal{V}^{(I)}. \quad (5)$$

Then, if we define $\gamma_{ij} = V_i V_j B_{ij}$, $i, j \in \mathcal{V}$, and differentiate (1) for all nodes i, we obtain that

$$\dot{P}(\theta(t)) = L^{(p)}(\theta(t)) \dot{\theta}(t), \quad (6)$$

where $L^{(p)}(\theta(t))$ is defined as follows: $L^{(p)}_{ij}(\theta(t)) = -\gamma_{ij} \cos(\theta_i(t) - \theta_j(t))$, $i \neq j$, $i, j \in \mathcal{V}$, and $L^{(p)}_{ii}(\theta(t)) = \sum_{l \neq i, l \in \mathcal{V}} \gamma_{il} \cos(\theta_i(t) - \theta_l(t))$. The matrix $L^{(p)}(\theta(t))$ can be partitioned as follows:

$$L^{(p)}(\theta(t)) = \begin{bmatrix} L^{(p)}_I(\theta(t)) & L^{(p)}_{IL}(\theta(t)) \\ L^{(p)}_{LI}(\theta(t)) & L^{(p)}_L(\theta(t)) \end{bmatrix}, \quad (7)$$

where $L^{(p)}_I(\theta(t)) \in \mathbb{R}^{|\mathcal{V}^{(I)}| \times |\mathcal{V}^{(I)}|}$ and $L^{(p)}_L(\theta(t)) \in \mathbb{R}^{|\mathcal{V}^{(L)}| \times |\mathcal{V}^{(L)}|}$. Note that since we assume ℓ_j to be constant, it follows from (4) that $\dot{P}_j(\theta(t)) = 0$, $\forall j \in \mathcal{V}^{(L)}$. Then, we can apply Kron reduction to (6) so as to eliminate the equations corresponding to load nodes, obtaining

$$\dot{P}^{(I)}(\theta(t)) = S(\theta(t)) \dot{\theta}^{(I)}(t), \quad (8)$$

where $P^{(I)}(\theta(t)) = [P_1(\theta(t)), \ldots, P_m(\theta(t))]^{\mathrm{T}}$, $\theta^{(I)}(t) = [\theta_1(t), \ldots, \theta_m(t)]^{\mathrm{T}}$, and $S(\theta(t))$ denotes the Schur complement of $L^{(p)}(\theta(t))$, i.e., $S(\theta(t)) = L^{(p)}_I(\theta(t)) - L^{(p)}_{IL}(\theta(t))(L^{(p)}_L(\theta(t)))^{-1}(\theta(t)) L^{(p)}_{LI}(\theta(t))$, where $L^{(p)}_L(\theta(t))$ is invertible for all t since the controller we propose in Section III enforces the so-called phase-cohesiveness property, i.e., $|\theta_i(t) - \theta_j(t)| \leq \epsilon$, $\forall \{i, j\} \in \mathcal{E}_p$, and $\epsilon \in [0, \frac{\pi}{2})$ [10]; $\theta(t)$ is referred to as phase-cohesive. Then, by combining (5) and (8), the dynamics of the microgrid can be described as follows:

$$\begin{aligned} \dot{\theta}^{(I)}(t) &= u(t), \\ \dot{P}^{(I)}(t) &= S(\theta(t)) u(t), \end{aligned} \quad (9)$$

where $u(t) = [u_1(t), \ldots, u_m(t)]^{\mathrm{T}}$.

Note that if $|\theta_i(t) - \theta_j(t)| < \frac{\pi}{2}$ at any given time t, $\forall \{i, j\} \in \mathcal{E}$, then, $L^{(p)}(\theta(t))$ is the weighted Laplacian of graph $\mathcal{G}$, and $S(\theta(t))$ is positive semidefinite with a single zero eigenvalue, the corresponding eigenvector of which being the all-ones vector, $\mathbb{1}_{\mathrm{I}}$. Also, the second smallest eigenvalue of $S(\theta(t))$, denoted by $\lambda_2(S(\theta(t)))$, is positive.

B. Control Objectives

Suppose that, before time $t = t_0$, some substantial load change occurs which results in (i) the frequency deviating from its nominal value, and (ii) a mismatch between the active power injections at the inverter nodes and the reference; then, the main control objectives are:

- O1 to track some desired injections, P_i^*, $i \in \mathcal{V}_p^{(I)}$, which satisfy: (P1) $\sum_{i \in \mathcal{V}_p^{(I)}} P_i^* = \sum_{l \in \mathcal{V}_p^{(L)}} \ell_l$, and (P2) ensure that the resultant equilibrium point is phase-cohesive; and
- O2 to eliminate the frequency deviation that results from the load change, i.e., $\dot{\theta}_i(t) \to 0$ as $t \to \infty$, $i \in \mathcal{V}_p$.

Inverters typically share the active power demand of the loads according to their power ratings $\frac{P_i^*}{P_j^*} = \frac{D_i}{D_j} = \frac{\overline{P}_i}{\overline{P}_j}$, $i, j \in \mathcal{V}^{(I)}$, where $\overline{P}_i$ is the power rating of inverter i [3]. [Later, we discuss how to pick the reference differently.]

For later developments, we also note that all stability-related issues and notions are meant with respect to the system model in (9), in which voltages are assumed to be constant, and the objectives O1 and O2.

III. Active Power Load Sharing Control

In this section, we first propose a control strategy to provide frequency regulation and to achieve the desired active power load sharing among the inverters. Then, under some assumptions, we show that this control ensures global convergence of the active power injections to the desired reference and eliminates the frequency error.

A. Controller Operation Description

Let P_i^* denote the desired power injected by inverter i, and assume it satisfies P1 and P2 as stated in objective O1. Then, if for the system in (9) we were to utilize a controller of the form

$$u(t) = -\alpha(P^{(I)}(t) - P^*), \tag{10}$$

where $P^* = [P_1^*, \ldots, P_m^*]^{\mathrm{T}}$ and $\alpha > 0$, it might be the case that $P^{(I)}(t)$ may not converge to P^* as $t \to \infty$. In fact, the system might become unstable, since the phase-cohesiveness property might be violated at some time and, as a result, the matrix $S(\theta(t))$ might not be always positive-semidefinite with a single zero eigenvalue. However, we can enforce the phase-cohesiveness property at all times if we modify the controller in (10) by replacing P^* with a piecewise constant reference signal, $P^r(t) = [P_1^r(t), \ldots, P_m^r(t)]^{\mathrm{T}}$, based on the following observation to be proved later: when, at some $t = \tau$, the reference $P^r(t)$ is fixed to some constant value, for $t \geq \tau$, within some $\delta > 0$ of the actual power injections at inverter nodes, i.e., $\|P^{(I)}(\tau) - P^r(\tau)\|_2 \leq \delta$, the controller

$$u(t) = -\alpha(P^{(I)}(t) - P^r(t)) \tag{11}$$

drives $P^{(I)}(t) - P^r(t) \to 0$ as $t \to \infty$, while maintaining the phase-cohesiveness property at all times. When $\|P^{(I)}(t) - P^r(t)\|_2$ becomes very small, we move the reference $P^r(t)$ closer to the desired injection value P^* along the line connecting initial active power injections $P(\theta(t_0))$ and the reference P^*, and apply control in (11). This slow readjustment of $P^r(t)$ continues until we reach P^*. We show later that picking $P^r(t)$ that way allows us to maintain the phase-cohesiveness property for all time t.

More formally, after a substantial load change occurs, we trigger the controller at $t = t_0$ by choosing the reference value, $P^r(t)$, for $t \geq t_0$, as follows:

$$P^r(t) = (1 - \lambda(t))P^{(I)}[t_0] + \lambda(t)P^*, \tag{12}$$

where, for $t = \tau$, $P^{(I)}[\tau] := P^{(I)}(\theta(\tau))$, and $\lambda(t) = \lambda(t_0^-) + \Delta\lambda$, with

$$\Delta\lambda = \frac{\delta - \underline{\delta}\sqrt{m}}{\|P^* - P^{(I)}[t_0]\|_2}, \tag{13}$$

and $\lambda(t_0^-) = 0$. When for some $t = t_1$, $P^{(I)}[t_1^-]$ is such that $\|P^{(I)}[t_1^-] - P^r(t_1^-)\|_\infty \leq \underline{\delta}$, we again increase $\lambda(t)$ by $\Delta\lambda$:

$$\lambda(t) = \lambda(t_1^-) + \Delta\lambda, \quad t_1 \leq t,$$

and use the expression in (12) to update the reference value. By denoting t_k as the reference update time for which $t_k^- = \arg_{t > t_{k-1}} \|P^{(I)}(\theta(t)) - P^r(t)\|_\infty \leq \underline{\delta}$, $P^r(t)$ can be expressed in a more general form as follows:

$$P^r(t) = (1 - \lambda(t))P^{(I)}[t_0] + \lambda(t)P^*, \quad t_k \leq t < t_{k+1}, \tag{14}$$

where

$$\lambda(t) = \lambda(t_k^-) + \Delta\lambda, \quad t_k \leq t < t_{k+1}. \tag{15}$$

In a more compact form, (14) – (15) can be rewritten as follows:

$$P^r(t) = P^r(t_k^-) + \Delta P^r, \quad t_k \leq t < t_{k+1}, \tag{16}$$

where $\Delta P^r = (\delta - \underline{\delta}\sqrt{m})\frac{P^* - P^{(I)}[t_0]}{\|P^* - P^{(I)}[t_0]\|_2}$, with ΔP_i^r denoting the i-th component of ΔP^r; thus, at each $t = t_k$, we move $P^r(t)$ closer to P^* along the line connecting $P^{(I)}[t_0]$ and P^*. [Note that if at some time t, $\|P^{(I)}(\theta(t)) - P^*\| < \delta$, then, we set $\lambda(t)$ to 1 and $P^r(t)$ to P^*.]

In order to preserve the phase-cohesiveness property for all $t \geq t_0$, $P(\theta(t))$ needs to satisfy the following synchronization condition for all $t \geq t_0$ [10]:

$$\|M^{\mathrm{T}} L^\dagger P(\theta(t))\|_\infty \leq \sin\epsilon, \tag{17}$$

where $L^\dagger$ is the pseudoinverse of $L = [L_{ij}]$, with $L_{ij} = -\gamma_{ij}$, $i \neq j$, $i, j \in \mathcal{V}$, and $L_{ii} = \sum_{l \neq i, l \in \mathcal{V}} \gamma_{il}$.

Define $P[t_k] := P(\theta(t_k))$ and $\ell = [\ell_{m+1}, \ldots, \ell_n]^{\mathrm{T}}$. Later, we show that the synchronization condition given in (17) is always satisfied by the controller in (11) provided $P[t_0]$ and $\begin{bmatrix} P^* \\ -\ell \end{bmatrix}$ satisfy a more strict version of the synchronization condition in (17), given by

$$\|M^{\mathrm{T}} L^\dagger P(\theta(t))\|_\infty \leq \kappa \sin\epsilon, \tag{18}$$

for some $\kappa \in (0, 1)$; specifically, in the next section, we will establish that we need to pick

$$\delta = \frac{1 - \kappa}{\eta} \sin\epsilon, \tag{19}$$

where $\eta = \|M^{\mathrm{T}} X\|_\infty$, where X is a submatrix of $L^\dagger$ formed from its first m columns, and

$$\kappa = \frac{1}{\sin\epsilon} \max\left\{ \|M^{\mathrm{T}} L^\dagger P[t_0]\|_\infty, \left\| M^{\mathrm{T}} L^\dagger \begin{bmatrix} P^* \\ -\ell \end{bmatrix} \right\|_\infty \right\}. \tag{20}$$

B. Stability Analysis

Here, we prove that, for a certain δ, the phase-cohesiveness property is always preserved by the controller (11) with the reference in (16), provided that $P[t_0]$ and $\begin{bmatrix} P^* \\ -\ell \end{bmatrix}$ satisfy (18). In order to prove this result, we need the result in the following lemma.

Lemma 1. Suppose $\theta(t_0)$ is phase-cohesive, and $P(\theta(t))$ is continuous and satisfies the synchronization condition in (17) for all $t \geq t_0$. Then, $\theta(t)$ is phase-cohesive, unique and continuous for all $t \geq t_0$.

Now, we state the main stability result.

Proposition 1. Suppose that $P[t_0]$ and $\begin{bmatrix} P^* \\ -\ell \end{bmatrix}$ satisfy (18), and $\theta(t_0)$ is phase-cohesive. Then, if δ satisfies (19), the controller in (11) with the reference in (16) ensures that the phase-cohesiveness property is maintained for all t, and $P^{(I)}(\theta(t)) \to P^*$ as $t \to \infty$.

In the next lemma, we provide an upper bound on the time it takes for the injections to converge to the desired reference within a small error bound.

Lemma 2. Let $t = T$ denote the time it takes for $P^{(I)}(\theta(t))$ to converge to P^* within a small error bound, i.e., $\|P^{(I)}(\theta(t)) - P^*\|_\infty \leq \underline{\delta}\sqrt{m}$. Assuming $t_k = \inf_t \arg_{t>t_{k-1}} \|P^{(I)}(\theta(t)) - P^r(t)\|_\infty \leq \underline{\delta}$, we have that

$$T \leq \frac{\|P^* - P^{(I)}[t_0]\|_2 + \underline{\delta}\sqrt{m}}{\alpha\lambda_2(L)(\delta - \underline{\delta}\sqrt{m})\cos(\epsilon)} \ln \frac{\delta}{\underline{\delta}\sqrt{m}}. \tag{21}$$

IV. Active Power Reference Assignment

In this section, we discuss how to choose the active power reference distributively for tree networks, as well as for those networks that have non-overlapping cycles (any pair of cycles does not share any common edge) so as to satisfy the phase-cohesiveness property.

A. Tree Networks

Let $P_i^{(0)}$ denote the reference signal to inverter $i \in \mathcal{V}^{(I)}$ as determined by the load power sharing criterion in objective O2, i.e., $P_i^* = P_i^{(0)} := \frac{D_i}{\sum_{j\in\mathcal{V}^{(I)}} D_j} \sum_{l\in\mathcal{V}^{(L)}} \ell_l$; next, we discuss an alternative to this reference choice. To this end, define the following quadratic optimization problem:

$$\min_{p,\phi} \; \|p - p^{(0)}\|_2^2 + \|\phi\|_2^2 \tag{22}$$

$$\text{subject to } p = M\Gamma\phi, \tag{23}$$

$$-\kappa_0 \leq \phi_{ij} \leq \kappa_0, \quad \forall\{i,j\} \in \mathcal{E}, \tag{24}$$

$$\underline{P}_i \leq p_i \leq \overline{P}_i, \quad \forall i \in \mathcal{V}, \tag{25}$$

which we refer to as QP1, where $\underline{P}_i = \overline{P}_i = -\ell_i$, $i \in \mathcal{V}^{(L)}$, the constraint (23) is a flow balance constraint from (2), (24) – (25) are the box constraints on the normalized flows and power injections at each node, $\kappa_0 < \sin\epsilon$ is a positive constant parameter, $p \in \mathbb{R}^n$, $p^{(0)} = [p_1^{(0)}, \ldots, p_n^{(0)}]^{\mathrm{T}}$ with $p_i^{(0)} = P_i^{(0)}$, $i \in \mathcal{V}^{(I)}$, and $p_i^{(0)} = -\ell_i$, $i \in \mathcal{V}^{(L)}$.

The solution of QP1, denoted by (p^*, ϕ^*), will be used to assign the reference P^*, i.e., $P^* = [p_1^*, \ldots, p_m^*]^{\mathrm{T}}$, for our controller in (11). By penalizing deviation of p from $p^{(0)}$ in the cost function of QP1 in (22), we aim to preserve the active power sharing among inverters according to their power ratings. The second term in the cost function allows us to minimize the flows along the electrical lines, which in general results in an increase of the step δ and improvement of the convergence rate of the controller in (11). Next, we show how to solve QP1 distributively.

First, we reformulate QP1 by introducing additional variables and constraints, the goal of which is to obtain an equivalent problem with a structure that is amenable to a solution via the alternating direction method of multipliers (ADMM); see, e.g. [12, Section 3.4]. To this end, introduce variables $\phi_{ij}^{(i)}$ and $\phi_{ij}^{(j)}$, and define $x^{(i)} = \{\phi_{ij}^{(i)}\}_{\{i,j\}\in\mathcal{E}}$ and $z^{(i)} = \{\phi_{ji}^{(i)}\}_{\{i,j\}\in\mathcal{E}}$. Then, clearly, QP1 is equivalent to the following optimization problem:

$$\begin{aligned} &\min_{p,x,z} \; \|p - p^{(0)}\|_2^2 + \|x\|_2^2 \\ &\text{subject to } \forall i \in \mathcal{V}, p_i = \sum_{j\in\mathcal{N}_i^+} \gamma_{ij}\phi_{ij}^{(i)} - \sum_{j\in\mathcal{N}_i^-} \gamma_{ij}\phi_{ji}^{(i)}, \\ &\qquad \underline{P}_i \leq p_i \leq \overline{P}_i, \\ &\qquad \forall\{i,j\} \in \mathcal{E}, -\kappa_0 \leq \phi_{ij}^{(i)} \leq \kappa_0, \\ &\qquad -\kappa_0 \leq \phi_{ij}^{(j)} \leq \kappa_0, \\ &\qquad \phi_{ij}^{(i)} = \phi_{ij}^{(j)}, \end{aligned}$$

which we refer to as QP2, where
$x = [(x^{(1)})^{\mathrm{T}}, \ldots, (x^{(n)})^{\mathrm{T}}]^{\mathrm{T}}$, $z = [(z^{(1)})^{\mathrm{T}}, \ldots, (z^{(n)})^{\mathrm{T}}]^{\mathrm{T}}$,

$$\mathcal{N}_i^+ = \{j : M_{je} = -1, e = \mathbb{I}(\{i,j\}), \{i,j\} \in \mathcal{E}\},$$

$$\mathcal{N}_i^- = \{j : M_{je} = 1, e = \mathbb{I}(\{i,j\}), \{i,j\} \in \mathcal{E}\}.$$

Then, the augmented Lagrangian for QP2 is given by

$$\begin{aligned} L_\rho(p,x,z,\mu,\nu) =& \|p - p^{(0)}\|_2^2 + \|x\|_2^2 \\ &+ \sum_{i=1}^m \mu_i(p_i - \sum_{j\in\mathcal{N}_i^+} \gamma_{ij}\phi_{ij}^{(i)} + \sum_{j\in\mathcal{N}_i^-} \gamma_{ij}\phi_{ji}^{(i)}) \\ &+ \sum_{i=1}^m \frac{\rho}{2}(p_i - \sum_{j\in\mathcal{N}_i^+} \gamma_{ij}\phi_{ij}^{(i)} + \sum_{j\in\mathcal{N}_i^-} \gamma_{ij}\phi_{ji}^{(i)})^2 \\ &+ \sum_{\{i,j\}\in\mathcal{E}} \Big(\nu_{ij}(\phi_{ij}^{(i)} - \phi_{ij}^{(j)}) + \frac{\rho}{2}(\phi_{ij}^{(i)} - \phi_{ij}^{(j)})^2\Big), \end{aligned}$$

where $\rho > 0$ is some constant, $\mu = [\mu_1, \ldots, \mu_n]^{\mathrm{T}}$ and $\nu = [\nu_1, \ldots, \nu_{|\mathcal{E}|}]^{\mathrm{T}}$. To solve QP2, we apply ADMM, the iterations of which are as follows:

$$\begin{bmatrix} p[k+1] \\ x[k+1] \end{bmatrix} = \min_{(p,x)\in\mathcal{D}} L_\rho(p,x,z[k],\mu[k],\nu[k]), \tag{26}$$

$$z[k+1] = \min_{z\in\mathcal{D}'} L_\rho(p[k+1],x[k+1],z,\mu[k],\nu[k]), \tag{27}$$

$$\begin{aligned} \mu_i[k+1] =& \mu_i[k] + \rho(p_i[k+1] - \sum_{j\in\mathcal{N}_i^+} \gamma_{ij}\phi_{ij}^{(i)}[k+1] \\ &+ \sum_{j\in\mathcal{N}_i^-} \gamma_{ij}\phi_{ji}^{(i)}[k+1]), \end{aligned} \tag{28}$$

$$\nu_{ij}[k+1] = \nu_{ij}[k] + \rho(\phi_{ij}^{(i)}[k+1] - \phi_{ij}^{(j)}[k+1]), \tag{29}$$

for all $i \in \mathcal{V}$ and $\{i,j\} \in \mathcal{E}$, where

$$\begin{aligned} \mathcal{D} = \{(p,x) :\; & \underline{P}_j \leq p_j \leq \overline{P}_j, \forall j \in \mathcal{V}, -\kappa_0 \leq \phi_{ij}^{(i)} \leq \kappa_0, \\ & \forall\{i,j\} \in \mathcal{E}\}, \end{aligned}$$

$$\mathcal{D}' = \{z : -\kappa_0 \leq \phi_{ji}^{(i)} \leq \kappa_0, \forall (j,i) \in \mathcal{E}\}.$$

The iterations in (26) – (27) can be written for each node i as follows:

$$\begin{aligned} \begin{bmatrix} p_i[k+1] \\ x^{(i)}[k+1] \end{bmatrix} =& \min_{(p_i,x^{(i)})\in\mathcal{D}_i} (p_i - p_i^{(0)})^2 + \|x^{(i)}\|_2^2 + \mu_i[k]p_i \\ &- \mu_i[k] \sum_{j\in\mathcal{N}_i^+} \gamma_{ij}\phi_{ij}^{(i)} + \frac{\rho}{2}(p_i - \sum_{j\in\mathcal{N}_i^+} \gamma_{ij}\phi_{ij}^{(i)} + \sum_{j\in\mathcal{N}_i^-} \gamma_{ij}\phi_{ji}^{(i)}[k])^2 \\ &+ \sum_{j\in\mathcal{N}_i^+} \Big(\nu_{ij}[k]\phi_{ij}^{(i)} + \frac{\rho}{2}(\phi_{ij}^{(i)} - \phi_{ij}^{(j)}[k])^2\Big), \end{aligned} \tag{30}$$

$$z^{(i)}[k+1] = \min_{z^{(i)} \in \mathcal{D}'_i} \mu_i[k] \sum_{j \in \mathcal{N}_i^-} \gamma_{ij}\phi_{ji}^{(i)} - \sum_{j \in \mathcal{N}_i^-} \nu_{ji}[k]\phi_{ji}^{(i)}$$
$$+ \frac{\rho}{2}(p_i[k+1] - \sum_{j \in \mathcal{N}_i^+} \gamma_{ij}\phi_{ij}^{(i)}[k+1] + \sum_{j \in \mathcal{N}_i^-} \gamma_{ij}\phi_{ji}^{(i)})^2$$
$$+ \sum_{j \in \mathcal{N}_i^-} \frac{\rho}{2}(\phi_{ji}^{(j)}[k+1] - \phi_{ji}^{(i)})^2, \quad (31)$$

where

$$\mathcal{D}_i = \{(p_i, x^{(i)}) : \underline{P}_i \leq p_i \leq \overline{P}_i, -\kappa_0 \leq \phi_{ij}^{(i)} \leq \kappa_0, \forall j \in \mathcal{N}_i^+\},$$

and $\mathcal{D}'_i = \{z^{(i)} : -\kappa_0 \leq \phi_{ji}^{(i)} \leq \kappa_0, \forall j \in \mathcal{N}_i^-\}$.

The iterations in (28) – (31) can be executed in a distributive fashion whereby each node i only needs to obtain $\phi_{ji}^{(j)}[k+1]$ from every $j \in \mathcal{N}_i^-$, and $\phi_{ij}^{(j)}[k+1]$ from every $j \in \mathcal{N}_i^+$ by communicating with its neighbors. If QP1 is feasible, then, by [13, Theorem 1], (26) – (29) converge to the optimal phase-cohesive solution $(p^*, \phi^*, \phi^*, \mu^*, \nu^*)$. However, solving QP1 does not necessarily yield a phase-cohesive solution if there are cycles in the network because there might not exist a phase-cohesive θ^* corresponding to (p^*, ϕ^*) satisfying

$$\sum_{\{l,j\} \in \mathcal{C}_i} (\theta_l^* - \theta_j^*) = 0, \quad (32)$$

where $\mathcal{C}_i$ is an oriented cycle in the network. Next, we show how to deal with the case when the network has non-overlapping cycles. To this end, we add an additional constraint to QP1 which guarantees phase-cohesiveness for the cyclic networks.

B. Networks with Non-Overlapping Cycles

Consider a network with c non-overlapping cycles denoted by $\mathcal{C}_1, \ldots, \mathcal{C}_c$. Define $N = [n^{(1)}, \ldots, n^{(c)}]$, where $n^{(i)}$ satisfies $Mn^{(i)} = 0$ and corresponds to cycle $\mathcal{C}_i$ so that, for $k = \mathbb{I}(\{l,j\})$,

$$n_k^{(i)} = \begin{cases} 1 & \text{if } \{l,j\} \in \mathcal{C}_i, \\ -1 & \text{if } \{j,l\} \in \mathcal{C}_i, \\ 0 & \text{else.} \end{cases}$$

Then, for a given set of injections p, any solution of (2) can be generally written as follows [10]:

$$\phi' = \phi + \Gamma^{-1}N\mu, \quad (33)$$

where ϕ is any particular solution to $p = M\Gamma\phi$, and $\mu \in \mathbb{R}^c$.

Assuming $\phi_{ij} = -\phi_{ji}$, $\forall\{i,j\} \in \mathcal{E}_p$, define

$$\overline{\mu}_i(\phi) = \min_{\{l,j\} \in \mathcal{C}_i} \gamma_{lj}(\kappa_0 - \phi_{lj}), \quad (34)$$
$$\underline{\mu}_i(\phi) = \max_{\{l,j\} \in \mathcal{C}_i} \gamma_{lj}(-\kappa_0 - \phi_{lj}),$$

and $g_i(\phi) = \sum_{\{l,j\} \in \mathcal{C}_i} \arcsin(\phi_{lj})$. In order for p to have a corresponding phase-cohesive θ, we must show that there exists a $\mu \in \mathbb{R}^c$ such that $\phi + \Gamma^{-1}N\mu$ satisfies the box constraints in (24) and the following constraint:

$$g_i(\phi + \Gamma^{-1}n^{(i)}\mu_i) = 0, \quad i = 1, \ldots, c, \quad (35)$$

which is equivalent to satisfying the constraint in (32). By using the single cycle feasibility lemma in [10], the following result becomes obvious.

Lemma 3. Suppose (p, ϕ) satisfies all constraints of QP1 in (23) – (25). Then, there exists a $\mu \in \mathbb{R}^c$ such that $\phi + \Gamma^{-1}N\mu$ satisfies the box constraints in (24) and the constraint in (35) if

$$g_i(\phi + \Gamma^{-1}n^{(i)}\overline{\mu}_i(\phi)) \geq 0, \text{ and} \quad (36)$$
$$g_i(\phi + \Gamma^{-1}n^{(i)}\underline{\mu}_i(\phi)) \leq 0, \quad i = 1, 2, \ldots, c. \quad (37)$$

Because the constraints in (36) – (37) are non-convex, we obtain their inner convex approximation in the next developments.

For some $\beta \geq 0$, define

$$\mathcal{F}_i(\beta) = \{\phi : g_i(\phi + \Gamma^{-1}n^{(i)}\overline{\mu}_i(\phi)) \geq 0, \ \overline{\mu}_i(\phi) \geq \beta,$$
$$g_i(\phi + \Gamma^{-1}n^{(i)}\underline{\mu}_i(\phi)) \leq 0, \ \underline{\mu}_i(\phi) \leq -\beta\}.$$

Now, we state the following lemma.

Lemma 4. Suppose d_i is the number of edges in cycle $\mathcal{C}_i$, $\overline{\gamma}_i = \max_{\{l,j\} \in \mathcal{C}_i} \gamma_{lj}\kappa_0$, $\underline{\gamma}_i = \min_{\{l,j\} \in \mathcal{C}_i} \gamma_{lj}\kappa_0$, $\epsilon_0 = \arcsin(\kappa_0)$, and $\psi_i = \frac{\epsilon_0}{d_i - 1}$. If

$$\beta_i^* = \frac{1}{2}\overline{\gamma}_i - \frac{\underline{\gamma}_i}{2}\sin(\psi_i), \quad (38)$$

then, $\mathcal{F}_i(\beta_i^*) \equiv \mathcal{B}_i$, where

$$\mathcal{B}_i = \{\phi : -\kappa_0 + \frac{\beta_i^*}{\gamma_{lj}} \leq \phi_{lj} \leq \kappa_0 - \frac{\beta_i^*}{\gamma_{lj}}, \forall\{l,j\} \in \mathcal{C}_i\}.$$

By Lemma 4, if $\phi \in \mathcal{F}_i(\beta_i^*)$, then, clearly ϕ also satisfies the cycle constraints in (36) – (37); therefore, by Lemma 3, there exists a $\mu \in \mathbb{R}^c$ such that $\phi + \Gamma^{-1}N\mu$ satisfies the box constraints in (24) and the constraint in (35).

For each cycle $\mathcal{C}_i$, $i = 1, \ldots, c$, in the network, choose β_i^* as in (38); then, we modify QP2 by enforcing additional constraints of having $\phi \in \mathcal{B}_i$, $i = 1, \ldots, c$:

$$\min_{p,x,z} \|p - p^{(0)}\|_2^2 + \|x\|_2^2$$
$$\text{subject to } \forall i \in \mathcal{V}, p_i = \sum_{j \in \mathcal{N}_i^+} \gamma_{ij}\phi_{ij}^{(i)} - \sum_{j \in \mathcal{N}_i^-} \gamma_{ij}\phi_{ji}^{(i)},$$
$$\underline{P}_i \leq p_i \leq \overline{P}_i,$$
$$\forall\{i,j\} \in \mathcal{E}, -\kappa_0 \leq \phi_{ij}^{(i)} \leq \kappa_0,$$
$$-\kappa_0 \leq \phi_{ij}^{(j)} \leq \kappa_0, \phi_{ij}^{(i)} = \phi_{ij}^{(j)},$$
$$x, z \in \mathcal{B}_k, \quad k = 1, \ldots, c,$$

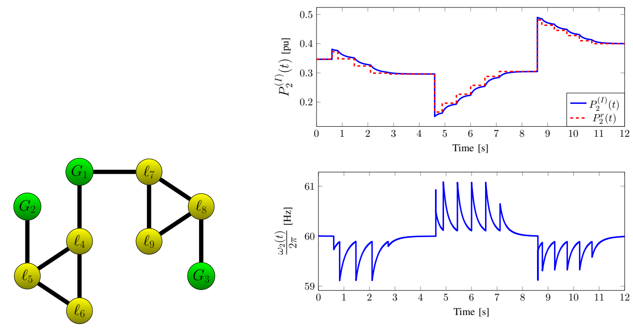
which we refer to as QP3. Then, clearly as discussed above, by Lemmas 3 – 4, the reference p^*, which results from solving QP3, has a corresponding phase-cohesive θ^*, which

TABLE I: Simulation Test Model Parameters.

$[P_{base}, \overline{P}_1, \overline{P}_2, \overline{P}_3]$, $\frac{1}{2\pi}\Delta\omega_{max}$	$[4, 6, 3, 3]$ MW, 1 Hz
$[B_{base}, B_{14}, B_{25}, B_{38}, B_{17}, B_{45}]$	$[4, 3, 3, 3, 2.5, 1.2]$ Ω^{-1}
$[B_{46}, B_{56}, B_{78}, B_{89}, B_{79}]$	$[1.3, 1, 1.4, 1, 1.2]$ Ω^{-1}
$[V_{base}, V_i, \forall i \in \mathcal{V}^{(I)}, V_l, \forall l \in \mathcal{V}^{(L)}]$	$[1, 1, 0.96]$ kV

TABLE II: Comparison between P^* and $P^{(0)}$ after a load change at $t = 0.6$ s.

	$\|\epsilon\|$	$\|\phi\|$	κ	δ	$\Delta\lambda$
P^*	$\frac{\pi}{2} - 0.01$	1.8388	0.7782	0.0278	0.2623
$P^{(0)}$	$\frac{\pi}{2} - 0.01$	2.3308	0.8090	0.0239	0.2410



(a) (b)

Fig. 1: (a) 9-bus microgrid topology, (b) $P_2^{(I)}(t)$ and $\frac{\omega_2(t)}{2\pi}$.

satisfies the cyclic constraints in (32). Define

$$\mathcal{H}_i = \{(p_i, x^{(i)}) : \underline{P}_i \leq p_i \leq \overline{P}_i, -\kappa_0 \leq \phi_{ij}^{(i)} \leq \kappa_0, \forall j \in \mathcal{N}_i^+, x^{(i)} \in \mathcal{B}_k, \forall k\},$$

$$\mathcal{H}_i' = \{z^{(i)} : -\kappa_0 \leq \phi_{ji}^{(i)} \leq \kappa_0, \forall j \in \mathcal{N}_i^-, z^{(i)} \in \mathcal{B}_k, \forall k\}.$$

The ADMM iterations for QP3 are the same as in (28) – (31) except that instead of the sets $\mathcal{D}_i$ and $\mathcal{D}_i'$, $\mathcal{H}_i$ and $\mathcal{H}_i'$ are used as constraint sets for $(p_i, x^{(i)}, z^{(i)})$. Since $\mathcal{H}_i$ and $\mathcal{H}_i'$ are merely box constraints, then, projection onto each of these sets is easy to compute.

V. Numerical Example

Consider a 9-bus lossless microgrid, the topology of which is shown in Fig. 1a. Nodes G_1, G_2, and G_3 correspond to inverters, with the remaining nodes corresponding to loads. The model parameters of the network are given in Table I. Maximum allowable frequency deviation $\frac{\Delta\omega_{max}}{2\pi}$ was chosen to be 1 Hz. Figure 1 shows the performance of the proposed control strategy for large load perturbations at $t = 0.6$ s, 4.6 s and 8.6 s. After each load perturbation, we are able to achieve the desired active power load sharing and regulate the frequency $\frac{\omega_i}{2\pi}$ at each inverter to the desired frequency value of 60 Hz. In order not to violate the maximum allowable frequency deviation, the upper bound on the feedback gain α was calculated as follows: $\alpha \leq \frac{\Delta\omega_{max}}{\delta} = \frac{2\pi}{\delta}$.

Controller parameters are given in Table II. Since $\kappa < 1$ throughout the whole operation of the controller, P^* was always feasible by the synchronization condition given in (18). We also chose $\underline{\delta} = 0.005$ such that $\underline{\delta}\sqrt{m} < \delta$ for $m = 3$. Because QP2-based reference P^* minimizes the flows, the normalized flow ϕ and κ are typically smaller for P^* than for $P^{(0)}$ as also shown in Table II, which further gives larger δ and $\Delta\lambda$ and improves the convergence rate. As implied by (21), it is desirable to have larger $\Delta\lambda$ and smaller κ in order to increase the convergence rate. The convergence speed can also be improved if we increase $\underline{\delta}$, while satisfying $\underline{\delta}\sqrt{m} < \delta$, as established in (21) and confirmed by the numerical simulations.

VI. Concluding Remarks

We presented a control strategy which allows inverter-interfaced generators in a microgrid to track globally some desired active power reference, and to restore the frequency to its nominal value by ensuring that the phase-cohesiveness property is always maintained. Even when large load perturbations initially set the actual active power injections far from the desired references, the proposed controller always achieves tracking if the initial injections and the references strictly satisfy the phase synchronization condition.

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