On Input-to-State Stability Notions for Reachability Analysis of Power Systems

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***Abstract*—We discuss the concept of input-to-state stability in dynamical systems and its application to address the reachability problem in power systems, i.e., what are the bounds on all possible system trajectories given certain parameters are unknown but bounded. We illustrate the application of the concept with a single-machine infinite-bus system where the infinite bus voltage is not perfectly known except for an upper and lower bound.**

I. Introduction

The concept of input-to-state stability (ISS) was introduced by Sontag in [1] and has been extensively studied and applied in systems and control theory over the last two decades (see, e.g., [2] and the references therein). It provides a method to determine whether or not the state x of a dynamical system of the form $\dot{x} = f(x, u)$ will remain bounded over time given the system input u is bounded. In a linear system $\dot{x} = Ax + Bu$, asymptotic stability, i.e., for $u = 0$, all the eigenvalues of A have negative real parts, ensures ISS; however this might not be the case in a non-linear system, i.e., asymptotic stability of the system with zero input $\dot{x} = f(x, 0)$ does not guarantee ISS of the non-zero input system $\dot{x} = f(x, u)$. The problem of asymptotic stability of power systems has been widely studied in the context of transient stability analysis, which is concerned with the ability of the system to maintain synchronism after a severe system disturbance [3], [4], [5], [6]. To the authors knowledge, the problem of ISS has not been addressed in the power system literature before.

In power systems analysis, ISS notions are of interest to understand the impact on dynamic performance of uncertainty caused by any uncontrolled and unpredictable change (not necessarily a fault) on the demand or on the supply side, e.g., generation based on intermittent renewable resources such as solar or wind. In this regard, we might be interested in assessing whether or not the frequency of the system, or the voltage at certain buses remain within certain limits given that the system is subject to uncontrolled disturbances. This problem can be addressed by computing the reach set or attainability domain, i.e., the set that bounds all possible trajectories given that the system initial conditions and input are restricted to some sets [7]. Computing the exact shape of the reach set can be a very difficult, or even impossible, task, especially for nonlinear systems. However, there is a close connection between ISS and reachability analysis and therefore we use tools developed in the context of ISS can be used to obtain bounds on the reach set.

The reachability problem in power systems subject to uncertainties has been addressed before in the context of power flow analysis, where the system model is a nonlinear algebraic equation (e.g., see [8], [9], [10] and the references therein). Reachability analysis tools have also been used in power systems transient stability analysis for computing the domain of attraction on an equilibrium point [11].

The paper is structured as follows. In section II, the concept of input-to-state stability (ISS) is formally introduced and a theorem providing a Lyapunov-like characterization of ISS is presented. In section III, we apply the ISS concept to a single-machine infinite-bus (SMIB) system. Section IV gives some concluding remarks.

II. Input-to-State Stability

Consider a nonlinear dynamical system of the form

$$\dot{x} = f(x, u), \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Suppose that $f(\cdot, \cdot)$ is locally Lipschitz in x and u and that the unforced system $\dot{x} = f(x, 0)$ has an asymptotically stable equilibrium point at the origin.

If the system (1) is forward complete, then the solution $x(t)$ exists for all $t \geq 0$ and it will be contained in some set $\mathcal{R}(t)$, which is called the reach set or attainability domain.

Definition 1: The system (1) is said to be *input-to-state stable* if there exist some functions $\gamma \in \mathcal{K}$[1] and $\beta \in \mathcal{KL}$[2] such that for every initial state $x_0 \in \mathbb{R}^n$, and each input $u(\cdot)$, the solution $x(t)$ of (1) satisfies

$$|x(t)| \leq \max\left(\beta(|x_0|, t), \gamma(\|u\|_{[0,t]})\right). \tag{2}$$

for all $t \geq 0$, where $\|\cdot\|_{[0,t]}$ denotes the supremum norm on the interval $[0, t]$. ■

The function β in (2) accounts for the the decaying influence of the initial condition x_0, whereas the funcion γ incorporates the influences of the input u. Furthermore, note that ISS

[1] A function α: $[0, \infty) \to [0, \infty)$ is of *class* $\mathcal{K}$ if α is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, it is of *class* $\mathcal{K}_\infty$.

[2] A function β: $[0, \infty) \times [0, \infty) \to [0, \infty)$ is of *class* $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$, and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$.

implies global asymptotic stability of the origin of the unforced system (i.e., with $u \equiv 0$), whereas in general, the converse is not true, as discussed in Section I. If (2) is only valid if x_0 and u are restricted to some sets $x_0 \in B_{x_0}$ and $u \in B_u$, then the system is said to be locally ISS.

In the context of this work, a Lyapunov-like ISS characterization given by the following theorem, the proof of which can be found in [12], will be helpful.

Theorem 1: Suppose there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathcal{K}$, a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ and a continuous, positive definite function $W : \mathbb{R}^n \to \mathbb{R}$ such that for all $x \in \mathbb{R}^n$

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \tag{3}$$

$$|x| \geq \rho(|u|) \;\Rightarrow\; \frac{\partial V}{\partial x} f(x,u) \leq -W(x). \tag{4}$$

Then the system (1) is ISS with

$$\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho. \tag{5}$$

Assumption (3) implies that the ISS-Lyapunov function V is positive definite ($V(0) = 0$ and $V(x) > 0$ for $x \neq 0$) and proper ($V(x) \to \infty$ as $|x| \to \infty$), and assumption (4) ensures that $\dot{V}$, the time derivative of V along trajectories of (1), is negative if the system state at time t, $x(t)$, lies outside a ball of radius $\rho(|u(t)|)$. We provide next a sketch of the proof of this theorem. A complete proof can be found in [12]. ■

The main idea behind the proof is that if $\dot{V}$ is negative outside a ball $B_\mu := \{x : |x| \geq \mu\}$, then, according to (3), $\dot{V}$ is negative everywhere on the level set $V(x) = \alpha_2(\mu)$ and thus the set $V(x) \leq \alpha_2(\mu)$ is invariant under (1). Therefore, using again (3), $|x|$ can be bounded from above by $\alpha_1^{-1}(\alpha_2(\mu))$. Taking $\mu = \rho(\|u\|_{[0,t]})$, we arrive at the result. If (4) is satisfied not for arbitrarily large x, but only for $x \in B_x$, where B_x is some subset of R^n including the origin, which is the case if the unforced system is only locally asymptotically stable, then we have to restrict u to be contained in some set B_u such that $\gamma(\|u\|_{[0,t]}) \in B_x$, and thus also x is contained in B_x. The system is then locally ISS if also $|x_0| \leq \gamma(\|u\|_{[0,t]})$.

A. Reachability Bounds

The connection between ISS and reachability analysis is straightforward and follows from Definition 1 and Theorem 1. Given the system (1), where the initial conditions and the input are only known to be contained in some sets B_{x_0} and B_u, respectively; then if the system is (locally) ISS, the reach set is bounded by some set $\Omega(t)$ defined by

$$\Omega(t) \;:=\; \left\{x : |x| \leq \max\left(\beta(|x_0|,t), \gamma(\|u\|_{[0,t]})\right)\right\}, \tag{6}$$

where $\gamma(\|u\|_{[0,t]}) = \alpha_1^{-1}(\alpha_2(\rho(\|u\|_{[0,t]})))$. For $t \gg 0$, the solution $x(t)$ will then be contained in a set $\mathcal{E}$, defined by

$$\mathcal{E} \;:=\; \left\{x : |x| \leq \alpha_1^{-1}\left(\alpha_2\left(\rho\left(u_{max}\right)\right)\right)\right\}, \tag{7}$$

where $u_{max} = \max_{u \in B_u} \|u\|$. Furthermore, if x_0 is small enough, i.e., x_0 is contained in some set $B_{x_0} := \{x : V(x) \leq \alpha_2(\rho(u_{max}))\}$, then the solution $x(t)$ will be contained in the set $\mathcal{E}$ for all $t \geq 0$.

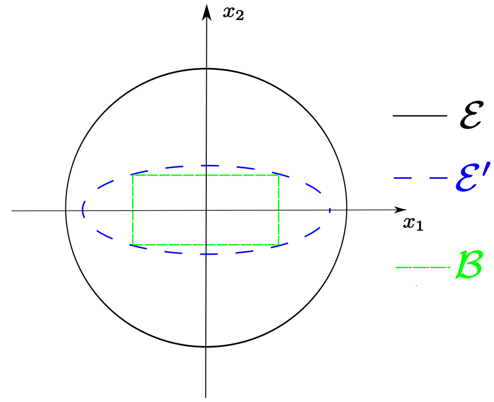

Fig. 1. Illustration of the sets $\mathcal{E}$, $\mathcal{E}'$ and $\mathcal{B}$.

Remark 1: A close examination of the proof of Theorem 1 shows that sometimes an even better bound for the reach set can be obtained. Namely, if $\dot{V}$ can be ensured to be negative outside a certain set $\mathcal{B}$ (not necessarily a ball as in Theorem 1), then the set

$$\mathcal{E}' := \{x : V(x) \leq V_{min}\}, \tag{8}$$

where V_{min} is the smallest possible value such that $\mathcal{B}$ is completely contained in $\mathcal{E}'$, is invariant with respect to (1), as $\dot{V}$ is negative on all its boundary, and therefore it is a better approximation for the reach set $\mathcal{R}(t)$ (for $t \gg 0$) than $\mathcal{E}$. Furthermore, if $x_0 \in \mathcal{E}'$, then the solution $x(t)$ will be contained in the set $\mathcal{E}'$ for all $t \geq 0$. However, the shape of the level sets of the ISS-Lyapunov function $V(x)$, and thus also the shape of $\mathcal{E}'$, might be complicated; thus it might be much easier to work with $\mathcal{E}$. An illustration of the sets $\mathcal{E}$, $\mathcal{E}'$ and $\mathcal{B}$ is given in Fig. 1. ■

III. Single-Machine Infinite-Bus System

We illustrate the application of ISS to quantify the effect of uncertainty in the infinite-bus voltage of a single-machine infinite-bus system (SMIB). Let δ be the angular position of the rotor in electrical radians, and ω_r be the angular velocity of the rotor in electrical rad/s. Then, the system can be described by the following state-space representation:

$$\begin{aligned}
\frac{d}{dt}\begin{bmatrix} \delta \\ \omega_r \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & -Da \end{bmatrix}\begin{bmatrix} \delta \\ \omega_r \end{bmatrix} + \begin{bmatrix} 0 \\ -ab\sin\delta \end{bmatrix} v_\infty \\
&\quad + \begin{bmatrix} 0 \\ a \end{bmatrix} T_m + \begin{bmatrix} -1 \\ Da \end{bmatrix} \omega_s \\
v_\infty \in \Omega_{v_\infty} &= \{v_\infty : |v_\infty - v_m| \leq k v_m\},
\end{aligned} \tag{9}$$

with $k > 0$, $v_m > 0$, $a = \frac{\omega_s}{2H}$ and $b = \frac{E_t}{X_m + X_l}$, where D, H, E_t, X_m, X_l, ω_s, and T_m are constant parameters [6].

For $v_\infty = v_m$, the (stable) equilibrium point of (9) is given by

$$\begin{aligned}
\omega_r^s &= \omega_s, \\
\delta^s &= \sin^{-1}\left(\frac{T_m}{b v_m}\right) \quad \in \left[0, \frac{\pi}{2}\right].
\end{aligned}$$

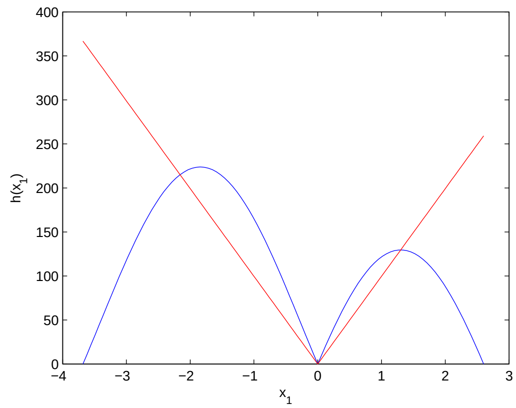

Fig. 2. $|h(x_1)|$ and its linear lower bound.

Applying the coordinate change $x_1 = \delta - \delta^s$, $x_2 = \omega_r - \omega_r^s$ in order to shift the equilibrium point to the origin and writing $v_\infty = v_m + \Delta v$, with $|\Delta v| \leq k v_m$ according to (9), results in the following system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -Dax_2 + aT_m - abv_m \sin(x_1 + \delta^s) \\ &\quad -ab\sin(x_1 + \delta^s)\Delta v \\ &=: -Dax_2 - h(x_1) - ab\sin(x_1 + \delta^s)\Delta v \end{aligned} \tag{10}$$

where

$$h(x_1) = abv_m \sin(x_1 + \delta^s) - aT_m. \tag{11}$$

Note that $x_1 h(x_1) > 0$ for all $x \in \mathcal{D} = \{x : -\pi - 2\delta^s < x_1 < \pi - 2\delta^s\}$ and

$$h_{max} := \max_{x \in \mathcal{D}} |h(x_1)| = \left|h\left(-\frac{\pi}{2} - \delta^s\right)\right| = a(T_m + bv_m). \tag{12}$$

Furthermore, for $x \in \overline{\mathcal{D}} = \{x : |x_1| \leq \frac{\pi}{2} - \delta^s\}$,

$$\begin{aligned} |h(x_1)| &\geq \frac{h(\frac{\pi}{2} - \delta^s)}{\frac{\pi}{2} - \delta^s}|x_1| = \frac{a\big(bv_m - T_m\big)}{\frac{\pi}{2} - \delta^s}|x_1| \\ &=: h_{app}|x_1|. \end{aligned} \tag{13}$$

A sketch of $|h(x_1)|$ is given in Fig. 2. Consider the following function $V(x)$ as an ISS-Lyapunov function for the system (10):

$$V(x) = \frac{1}{2}x'Px + \int_0^{x_1} h(y)dy \tag{14}$$

with

$$P = \begin{bmatrix} \epsilon a^2 D^2 & \epsilon a D \\ \epsilon a D & 1 \end{bmatrix}, \qquad 0 < \epsilon < 1.$$

According to (12), for all $x \in \mathcal{D}$,

$$V(x) \leq \frac{1}{2}x'Px + h_{max}|x_1| \leq \frac{1}{2}x'Px + h_{max}|x| \tag{15}$$

and thus $V(x)$ satisfies condition (3) of Theorem 1 with

$$\alpha_1(r) = \frac{1}{2}\lambda_{min}(P)r^2 \tag{16}$$

$$\alpha_2(r) = \left(\frac{1}{2}\lambda_{max}(P) + h_{max}\right) r^2, \tag{17}$$

where $\lambda_{min}(P)$ and $\lambda_{max}(P)$ denote the minimal (respectively, maximal) eigenvalue of the matrix P.

The derivative $\dot{V}$ is given by

$$\begin{aligned} \dot{V} &= \big(\epsilon a^2 D^2 x_1 + \epsilon a D x_2 + h(x_1)\big)x_2 \\ &\quad + \big(\epsilon a D x_1 + x_2\big)\big(-Dax_2 - h(x_1) \\ &\quad - ab\sin(x_1 + \delta^s)\Delta v\big) \\ &= -Da(1-\epsilon)x_2^2 - \epsilon a D x_1 h(x_1) \\ &\quad - ab\sin(x_1 + \delta^s)\big(\epsilon a D x_1 + x_2\big)\Delta v. \end{aligned} \tag{18}$$

This shows that the origin of the unforced system, i.e. for $\Delta v = 0$, is an asymptotically stable equilibrium point and $\mathcal{D}$ belongs to its region of attraction (in fact, it is the region of attraction), as $\dot{V}$ is negative for all $x \in \mathcal{D}$.

From (18), and using (13), we get that for all $x \in \overline{\mathcal{D}}$,

$$\begin{aligned} \dot{V} &\leq -Da(1-\epsilon)x_2^2 - \epsilon a D x_1 h(x_1) \\ &\quad + ab|\sin(x_1 + \delta^s)|\big(\epsilon a D|x_1| + |x_2|\big)|\Delta v| \\ &\leq -(1-\theta)Da(1-\epsilon)x_2^2 - (1-\theta)\epsilon h_{app} a D x_1^2 \\ &\quad -\theta Da(1-\epsilon)x_2^2 - \theta\epsilon h_{app} a D x_1^2 + \\ &\quad ab\big(\epsilon a D|x_1| + |x_2|\big)|\Delta v| \qquad 0 < \theta < 1 \\ &\leq -(1-\theta)Da(1-\epsilon)x_2^2 \\ &\quad -(1-\theta)\epsilon h_{app} a D x_1^2, \end{aligned} \tag{19}$$

if

$$\begin{aligned} |x_1| &\geq \frac{k_1 ab|\Delta v|}{h_{app}\theta} =: x_{1,min} \quad \text{or} \\ |x_2| &\geq \frac{k_2 b|\Delta v|}{D\theta(1-\epsilon)} =: x_{2,min} \end{aligned} \tag{20}$$

for large enough constants k_1 and k_2, which have to be chosen such that the terms in the fourth and fifth line of (19) are less than or equal to zero. Namely, if $|x_2| \geq x_{2,min}$, then the terms in the fourth and fifth line of (19) are less than or equal to

$$-\theta\epsilon h_{app} a D x_1^2 + a^2 b\epsilon D|x_1||\Delta v| - \frac{(k_2^2 - k_2)ab^2|\Delta v|^2}{D\theta(1-\epsilon)},$$

and if we choose k_2 large enough, this will be less or equal to zero for all x_1. In a similar way, we can determine a value for k_1 such that if $|x_1| \geq x_{1,min}$, then the terms in the fourth and fifth line of (19) are smaller or equal to zero for all x_2. Thus (19) holds if (20) is satisfied.

Next, note that (20) is implied by requiring that

$$|x| \geq \sqrt{x_{1,min}^2 + x_{2,min}^2}. \tag{21}$$

Therefore $V(x)$ satisfies condition (4) of Theorem 1 with

$$\rho(r) = r\sqrt{\left(\frac{k_1 ab}{\theta h_{app}}\right)^2 + \left(\frac{k_2 b}{D\theta(1-\epsilon)}\right)^2}. \tag{22}$$

Now suppose that the performance requirements are such that, first, we want to stay in the region $x \in \overline{\mathcal{D}}$ for which we established ISS; i.e., we want to have

$$|x_1| \leq \frac{\pi}{2} - \delta^s. \tag{23}$$

Second, we require that the electrical frequency deviations are smaller than 1 Hz, which is equivalent to requiring that the deviations in the machine speed have to be less than 2π rad/s, i.e.,

$$|x_2| \leq 2\pi. \tag{24}$$

If we work out a numerical example with typical parameter values, it turns out that the estimate for the reach set we get from (7) is too conservative. Namely, only for very small deviations in the input voltage, we can satisfy the performance requirements. For example, if we assume that $E_t = 1$ pu, X_m=0.2 pu, X_l=0.066 pu, H=4 pu, D=0.04 s/rad, T_m=1 pu, ω_s= 120π rad/s, v_m=1 pu, and $\epsilon = 0.3$, the performance requirements are met only for $k \leq 0.00035$, i.e., deviations in v_∞ must be smaller than 0.035% of its nominal value—a very conservative result.

However, if we use our knowledge of the shape of the level sets of the function $V(x)$, we get a better estimate. Note that these level sets have a shape close to an ellipsoid (in fact, the first term of (14) is an ellipsoid, and the second term squeezes this ellipsoid in the x_1-direction). According to Remark 1 in section II, we first compute the smallest level set $V(x) = V_{min}$ such that $\dot{V}$ is negative everywhere on it, which can be done by ensuring that all four edges of the rectangle defined by (20) lie inside this level set. This means that the level set $V(x) = V_{min}$ is invariant under (10) and thus the set $\mathcal{E}' = \{x : V(x) \leq V_{min}\}$ (8) is a better approximation for the reach set after the transient phase than the set $\mathcal{E}$ we get from (7). After that, we compute the largest level set $V(x) = V_{max,1}$ which completely lies inside the region specified by the first performance requirement (23). The maximum expansion of this level set in the x_1-direction is reached where the derivative of the implicit curve $\big(V(x) - V_{max,1}\big) = 0$ is parallel to the x_1-axis, i.e., where

$$\frac{\partial}{\partial x_2} V(x) = 0 \quad \Leftrightarrow \quad x_2 = -\epsilon a D x_1. \tag{25}$$

Thus $V_{max,1}$ can be calculated as

$$V_{max,1} = V\left(\frac{\pi}{2} - \delta^s, -\epsilon a D\left(\frac{\pi}{2} - \delta^s\right)\right). \tag{26}$$

Next, we compute the largest level set $V(x) = V_{max,2}$ that completely lies inside the region specified by the second performance requirement (24). The maximal expansion of this level set in the x_2-direction is reached where the derivative of $\big(V(x) - V_{max,2}\big) = 0$ is parallel to the x_2-axis:

$$\begin{aligned}\frac{\partial}{\partial x_1} V(x) = 0 \qquad \Leftrightarrow \qquad & \epsilon a^2 D^2 x_1 + \epsilon a D x_2 \\ & -aT_m + abv_m \sin(x_1 + \delta^s) = 0.\end{aligned}$$

Plugging in $x_2 = 2\pi$, according to (24), we can solve this numerically for x_1 (denote the result by $x_{1,v}$). Then,

$$V_{max,2} = V(x_{1,v}, 2\pi). \tag{27}$$

If $V_{max,2} < V_{max,1}$, then the second performance requirement is stricter (and vice versa). If both $V_{min} \leq V_{max,1}$ and $V_{min} \leq V_{max,2}$, then both performance requirements are met.

With the above given parameter values, it turns out that if $k \leq 0.03$, we get $V_{min} = 18.7$, which means that the set $\mathcal{E}' = \{x : V(x) \leq 18.7\}$ is an approximation for the reach set of the system (10). Furthermore, we obtain $V_{max,1} = 109.65$ and $V_{max,2} = 19.7$; this means that the constraint on the deviations in the electrical frequency (24) is much tighter than the constraint on x_1 (23) in order to remain in the ISS region. As $V_{min} < V_{max,2} < V_{max,1}$, both performance requirements are met. In fact, it turns out that if $k \leq 0.0308$, i.e., if the deviations in the input voltage V are smaller than 3.08%, then both performance requirements can be met—a significantly larger value than obtained before. If we further increase k beyond $k = 0.0308$, then $V_{min} > V_{2,max}$, and thus we cannot ensure that the second performance requirement is met.

IV. Concluding Remarks

In a real system, such frequency variations are not allowed because the speed governor keeps the frequency tight. The SMIB model used is very simplistic and does not include such control, which explains why such large frequency variation occurs even for small deviations in the magnitude of the infinite bus voltage. We chose such a simple model because the purpose of this paper is just to introduce the ISS concept to the power systems community and show its applications to quantify the effect of uncertainty in power system dynamics.

In general, the biggest challenge of the presented approach is to find a ”good” ISS-Lyapunov function such that the results are not too conservative or such that we know the shape of its level sets, as illustrated in the SMIB example. Further research needs to address the scalability of the method to multi-machine systems with higher-order machine dynamic models.

References

[1] E. Sontag, “Smooth stabilization implies coprime factorization,” *IEEE Transactions on Automatic Control*, vol. 34, no. 4, pp. 435–443, Apr 1989.

[2] ——, “Input to state stability: Basic concepts and results,” in *Nonlinear and Optimal Control Theory*, P. Nistri and G. Stefani, Eds. Berlin: Springer-Verlag, 2007, pp. 163–220.

[3] M. A. Pai., *Power System Stability*. New York, NY: North Holland Publishing Co., 1981.

[4] ——, *Energy Function Analysis for Power System Stability*. Boston, MA: Kluwer Academic Publishers, 1989.

[5] A. A. Fouad and V. Vittal., *Power System Transient Stability Analysis Using the Transient Energy Function Method*. Englewood Cliff, NJ: Prentice Hall, 1991.

[6] P. Sauer and A. Pai, *Power System Dynamics and Stability*. Upper Saddle River, NJ: Prentice Hall, 1998.

[7] F. Schweppe, *Uncertain Dynamic Systems*. Englewood Cliffs, NJ: Prentice-Hall Inc., 1973.

[8] F. Alvarado, Y. Hu, and R. Adapa, “Uncertainty in power system modeling and computation,” in *IEEE International Conference on Systems, Man and Cybernetics*, Oct 1992.

[9] Z. Wang and F. Alvarado, “Interval arithmetic in power flow analysis,” *IEEE Trans. Power Systems*, vol. 7, no. 3, pp. 1341–1349, Aug 1992.

[10] A. Saric and A. Stankovic, “Applications of ellipsoidal approximations to polyhedral sets in power system optimization,” *IEEE Trans. Power Systems*, vol. 23, no. 3, pp. 956–965, Aug. 2008.

[11] J. L. Jin, H. Liu, R. Kumar, V. Ajjarapu, J. McCalley, N. Elia, and V. Vittal, “An application of reachable set analysis in power system transient stability assessment,” in *Proc. Power Engineering Society General Meeting*, June 2005, pp. 1715–1719 Vol. 2.

[12] H. Khalil, *Nonlinear Systems*. Upper Saddle River, NJ: Prentice-Hall, 2002.