Price-Based Coordinated Aggregation of Networked Distributed Energy Resources

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***Abstract*—In this paper, we introduce a framework for studying the aggregated response for energy provision/consumption processes by distributed energy resources (DERs) that are physically connected to an electric power distribution network. In this framework, there is a set of agents referred to as aggregators that participate in a real-time energy market by submitting offers to sell, or bids to buy, a certain amount of energy at some price. To realize an offer or a bid, an aggregator interacts with a set of DERs and incentivizes them to produce (or consume) energy via some pricing strategy. In order to make a decision on whether or not to sell or buy energy, each DER uses the pricing information provided by the aggregator it is associated with, and some estimate of the average (or total) energy that neighboring DERs are willing to sell or buy, computed through some exchange of information among them through a cyber network; the topology of this cyber network is described by a connected undirected graph. The focus of this paper is on the DER strategic decision-making process, which we cast as a game with a single aggregator. In this context, we provide sufficient conditions on the aggregator's pricing strategy under which this game has a unique Nash equilibrium. Then, we propose a distributed algorithm that enables the DERs to seek this Nash equilibrium; this algorithm relies on simple computations using local information acquired through exchange of information with neighboring DERs. We illustrate our results through several numerical simulations.**

I. INTRODUCTION

Electric power systems are cyber-physical systems, where the functions of the electrically interconnected physical resources encompass one or more of generation, transmission, consumption of electrical energy, and the functions of the computational (cyber) resources are to monitor and control the entire system. In recent years, under the US DoE *Smart Grid* vision [2], and its European counterpart *Electricity Networks of the Future* [3], these cyber-physical systems have been undergoing radical transformations in structure and functionality. This is due to the integration of new renewable-based electricity generation resources (e.g., solar photovoltaics (PV) installations), and energy-storage capable loads (e.g., plug-in electric vehicles (PEVs)); and the increased reliance on advanced communications, which enables the active control of other types of energy-storage capable loads such as thermostatically-controlled loads (TCLs) (e.g., air conditioners, heat pumps, water heaters, and refrigerators).

These generation and controllable/storage-capable resources are commonly referred to as distributed energy resources (DERs) and, if properly coordinated, they provide new opportunities and more flexibility in the procurement of ancillary services such as frequency regulation and load following. For instance, PEVs and TCLs can be utilized to provide active power for up and down regulation services, e.g., energy peak-shaving during peak hours and load-leveling at night [4], [5], [6]. However, in order to enable the added functionality that these new technologies may provide, it is necessary to develop appropriate control mechanisms. In this paper, we address a particular instance of this problem and propose a framework for studying the aggregated response for energy provision/consumption processes by DERs in power distribution systems, with a focus on those DERs that have energy storage capabilities (e.g., PEVs and TCLs). While the motivation for this work was driven by electric power applications, the proposed framework might be useful in addressing similar problems that arise in other networked cyber-physical systems.

In our framework, we consider a set of aggregators, each of which participates in a real-time energy market[1] by submitting an offer to sell (or a bid to buy) a *block of energy*, i.e., a certain amount of energy over some period of time of fix duration (typically five minutes) [8], at some price. In order to determine the amount of energy to be sold or bought, each aggregator sends requests to a group of DERs to provide (or consume) energy and will incentivize them to do so via some pricing strategy, i.e., each aggregator offers to buy (or sell) energy via some pricing strategy; we assume that each DER group is determined beforehand and that different groups do not share members. This decoupling essentially allows us to concentrate on the interactions between a single aggregator and a single group of DERs associated with it.

After receiving a request from the aggregator, each DER will make a decision on whether or not it will provide (or consume) energy, and if it decides to do so, it will decide on the amount. The decision that each DER is faced with,

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Works of Gharesifard and Başar were supported in part by National Security Agency (NSA) through the Information Trust Institute of the University of Illinois at Urbana-Champaign, and in part by a grant from the Department of Energy. The work of Domínguez-García was supported in part by the National Science Foundation (NSF) under grant ECCS-CPS-1135598, and by the Consortium for Electric Reliability Technology Solutions (CERTS).

[1]This is a sub-hourly market, typically cleared every five minutes, which allows system operators to procure energy in real time after day-ahead and hour-ahead markets have run. The energy that operators procure through this market is used to balance instantaneous demand, reduce supply if demand falls, and curtail demand [7].

among other things, depends on its own utility function, along with a pricing strategy designed by the aggregator. Specifically, the algorithm we propose in the paper allows each DER to determine its decision, along with an estimate of the average (or total) amount of energy that the group of DERs decides to consume or produce; once this is done, each DER will drive its energy level to this value and the price the DERs get paid on (or will pay) is obtained by evaluating the corresponding pricing function, which depends on the average. In this context, the DERs are *price anticipating* in the sense that they are aware of the fact that the pricing is designed by the aggregator with respect to the average (or total) amount of energy that the group of DERs decides to consume or produce. We also assume that in order for each DER to make its decision, it can collect information from neighboring DERs with which it can exchange information.

The focus of this paper is on the DER decision-making process in response to a pricing strategy offered to them by an aggregator in order to incentivize them to produce or consume energy. We cast this decision-making process as a game and provide sufficient conditions on the aggregator's pricing strategy under which this game has a unique Nash equilibrium. Additionally, we propose a distributed algorithm that allows for the DERs to seek and converge to this Nash equilibrium; the algorithm relies on simple computations using local information acquired through exchange of information with neighboring DERs. We note that in the paper we do not address the problem of mechanism design that the aggregator may be faced with in order to incentivize DERs for any truth revelation. This would entail a different problem formulation than the one adopted here where the aggregator's pricing policy is fixed, and the focus is on the solution of the game at the DER level. In this sense, the problem under study here is a static problem. The mechanism design formulation would, however, constitute an interesting direction for future research.

A. Literature Review

The importance of distributed sensing and control in electrical energy systems has been discussed in several recent papers; examples of a vast literature include [4], [9], [10]. The distributed algorithm proposed in this work is related to distributed optimization algorithms for the optimization of a sum of convex functions (see e.g. [11], [12], [13], [14], [15], [16], [17]). All these works build on consensus-based dynamics to find the solutions to optimization problems in a variety of scenarios and are typically designed in discrete time, with the possible exception of [16], [17].

Game-theoretic models have been used recently for studying energy markets (see, e.g., [18], [19], [20], [21], [22]). The game-theoretic parts of our work are related to noncooperative resource allocation problems, see for example [23], [24], [25], where under appropriate concavity assumptions, the existence of a Nash equilibrium in pure strategies is guaranteed, using the results in [26]. Within the context of PEVs, the authors in [18] propose a game-theoretic model for studying their charging and discharging processes. In addition to the fact that the model does not take into account the original available charge of PEVs for participating in the game, the PEVs considered are not price anticipating, i.e., the model does not take into account the fact that the prices may be set based on the average available charge in the network. Also the fact that future PEVs are decision makers and have personal utility functions are not taken into account in this model. In the context of our work, the process of DER energy consumption is related to the charging process in PEVs that recently appeared in [21] and [27]; however, we also deal with a scenario in which the DERs are individual decision makers and arrive at the Nash equilibrium using the information available from their neighboring DERs (along with the price set by the aggregator). Thus, a key here is the role played by the cyber infrastructure for communications between DERs. Additionally, in our setting, we include more general pricing strategies and allow for analysis of scenarios in which DERs are capable of both providing and consuming energy.

B. Summary of Contributions

The first contribution of our work is the introduction of a distributed control framework for enabling the utilization of DERs (with energy storage capabilities) to provide load-following services by coordinating the amount of energy they provide (or consume). Our second contribution is the casting of the underlying competitive decision-making process as a multi-layer game and providing conditions on the aggregator's pricing strategy under which this game has a unique Nash equilibrium. Our third contribution is the design of a distributed iterative algorithm through which the DERs can arrive at the Nash equilibrium (when unique) of the game describing their decision-making process once the pricing strategies for energy provision/consumption are set. We establish the asymptotic convergence property of this distributed algorithm when the payoff functions are locally Lipschitz (i.e., not necessarily differentiable) and concave, and the underlying DERs' network is undirected and connected. As a by-product, our distributed scheme can be used for inducing the Nash equilibrium, when unique, for other locally Lipschitz concave games on undirected graphs with no shared constraints.

C. Organization

The remainder of this paper is organized as follows. Section II provides some of the mathematical background needed in subsequent developments. In Section III, we introduce our proposed market model and cast the behavior of its participants as a game. In Section IV, we analyze the existence and uniqueness of equilibrium points of the aforementioned game. In Section V, we propose a distributed algorithm that enables market participants to seek the equilibrium. Section V presents case studies that illustrate performance of the algorithm under different scenarios. Concluding remarks and directions for future work are presented in Section VII.

II. Mathematical Preliminaries

We start with some notational conventions. Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$, and $\mathbb{Z}_{\geq 1}$ denote the set of real, nonnegative real, integer, and positive integer numbers, respectively. We denote by $\mathscr{B}(\mathcal{A})$ the set of bounded real-valued functions on a set $\mathcal{A} \subset \mathbb{R}^d$, $d \in \mathbb{Z}_{\geq 1}$; we use $\mathscr{B}^0(\mathcal{A})$ when the functions are, additionally, continuous. We denote by $\mathrm{co}(\mathcal{A})$ the convex hull and by $\mathring{\mathcal{A}}$ the interior of A. We use the short-hand notation $\mathbf{1}_d = (1, \ldots, 1)^T \in \mathbb{R}^d$ and $\mathbf{0}_d = (0, \ldots, 0)^T \in \mathbb{R}^d$.

A. Nonsmooth Analysis

We recall some notions from nonsmooth analysis [28]. A function $f : \mathbb{R}^d \to \mathbb{R}$ is *locally Lipschitz* at $x \in \mathbb{R}^d$ if there exists a neighborhood $\mathcal{U}$ of x and $C_x \in \mathbb{R}_{\geq 0}$ such that $|f(y) - f(z)| \leq C_x ||y - z||$, for $y, z \in \mathcal{U}$; f is locally Lipschitz on $\mathbb{R}^d$ if it is locally Lipschitz at x for all $x \in \mathbb{R}^d$. A locally Lipschitz f is differentiable almost everywhere and its *generalized gradient* is

$$\partial f(x) = \mathrm{co}\big\{ \lim_{k\to\infty} \nabla f(x_k) \mid x_k \to x, x_k \notin \Omega_f \cup S \big\},$$

where Ω_f is the set of points where f fails to be differentiable and S is any set of measure zero. We recall the following properties of generalized gradients [28].

Lemma 2.1: **(Continuity of the generalized gradient map):** Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function at $x \in \mathbb{R}^d$. Then, the set-valued map $\partial f : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is upper semicontinuous and locally bounded at $x \in \mathbb{R}^d$ and moreover, $\partial f(x)$ is nonempty, compact, and convex.

For $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $z \in \mathbb{R}^d$, we let $\partial_x f(x, z)$ denote the generalized gradient of $x \mapsto f(x, z)$. Similarly, for $x \in \mathbb{R}^d$, we let $\partial_z f(x, z)$ denote the generalized gradient of $z \mapsto f(x, z)$. A point $x \in \mathbb{R}^d$ with $\mathbf{0}_d \in \partial f(x)$ is a *critical point* of f. A function $f : \mathbb{R}^d \to \mathbb{R}$ is *regular* at $x \in \mathbb{R}$ if for all $v \in \mathbb{R}^d$ the right directional derivative of f, in the direction of v, exists at x and coincides with the generalized directional derivative of f at x in the direction of v. We refer the reader to [28] for definitions of these notions. A convex and locally Lipschitz function at x is regular [28, Proposition 2.3.3]. The notion of regularity plays an important role when considering sums of Lipschitz functions as the next result shows.

Lemma 2.2: **(Finite sum of locally Lipschitz functions):** Let $\{f^i\}_{i=1}^n$ be locally Lipschitz at $x \in \mathbb{R}^d$. Then $\partial(\sum_{i=1}^n f^i)(x) \subseteq \sum_{i=1}^n \partial f^i(x)$, and equality holds if f^i is regular for $i \in \{1, \ldots, n\}$.

Here the summation on the righthand-side of the inequality should be understood in the sense described in [28]. A locally Lipschitz and convex function f satisfies, for all $x, x' \in \mathbb{R}^d$ and $\xi \in \partial f(x)$, the *first-order condition* of convexity, $f(x') - f(x) \geq \xi \cdot (x' - x)$.

B. Set-valued dynamical systems

Here, we recall some background on set-valued dynamical systems following [29]. A continuous-time set-valued dynamical system on $\mathsf{X} \subset \mathbb{R}^d$ is a differential inclusion

$$\dot{x}(t) \in \Psi(x(t)) \tag{1}$$

where $t \in \mathbb{R}_{\geq 0}$ and $\Psi : \mathsf{X} \subset \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is a set-valued map (the double arrow notation is used to show that points in X are mapped to subsets of $\mathbb{R}^d$). A solution to this dynamical system is an absolutely continuous curve $x : [0, T] \to \mathsf{X}$ which satisfies (1) almost everywhere. The set of equilibria of (1) is denoted by $\mathrm{Eq}(\Psi) = \{x \in \mathsf{X} \mid 0 \in \Psi(x)\}$. A sufficient condition for the existence of a solution is presented next, see [30, Chapter 2, Theorem 1].

Lemma 2.3: **(Existence of solutions):** For $\Psi : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ upper semicontinuous with nonempty, compact, and convex values, there exists a solution to (1) from any initial condition.

The LaSalle Invariance Principle is helpful to establish the asymptotic convergence of systems of the form in (1). A set $W \subset \mathsf{X}$ is *weakly positively invariant* under (1) if, for each $x \in W$, there exists at least one solution of (1) starting from x entirely contained in W. Similarly, W is *strongly positively invariant* under (1) if, for each $x \in W$, all solutions of (1) starting from x are entirely contained in W. Finally, the *set-valued Lie derivative* of a differentiable function $V : \mathbb{R}^d \to \mathbb{R}$ with respect to Ψ at $x \in \mathbb{R}^d$ is $\widetilde{\mathcal{L}}_\Psi V(x) = \{v^T \nabla V(x) \mid v \in \Psi(x)\}$.

Theorem 2.4: **(Set-valued LaSalle Invariance Principle):** Let $W \subset \mathsf{X}$ be strongly positively invariant under (1) and $V : \mathsf{X} \to \mathbb{R}$ be a continuously differentiable function. Suppose the evolutions of (1) are bounded and $\max \widetilde{\mathcal{L}}_\Psi V(x) \leq 0$ or $\widetilde{\mathcal{L}}_\Psi V(x) = \emptyset$, for all $x \in W$. Let $S_{\Psi,V} = \{x \in \mathsf{X} \mid 0 \in \widetilde{\mathcal{L}}_\Psi V(x)\}$. Then any solution $x(t)$, $t \in \mathbb{R}_{\geq 0}$, starting in W converges to the largest weakly positively invariant set M contained in $\bar{S}_{\Psi,V} \cap W$. When M is a finite collection of points, then the limit of each solution equals one of them.

C. Graph Theory

A *directed graph*, or simply *digraph*, is a pair $\mathcal{G} = (V, E)$, where V is a finite set called the vertex set and $E \subseteq V \times V$ is the edge set. We call $\mathcal{G}$ an *undirected graph* or simply a *graph* if, whenever $(i, j) \in E$, then also $(j, i) \in E$. In this paper, we only deal with undirected graphs. Given an edge $(u, v) \in E$, we call u and v neighbors and denote the set of neighbors of v by $\mathcal{N}_\mathcal{G}(v)$. A graph is called *connected* if there exists a path between any two vertices. A *weighted graph* is a triplet $\mathcal{G} = (V, E, \mathsf{A})$, where (V, E) is a graph and $\mathsf{A} \in \mathbb{R}_{\geq 0}^{n\times n}$ is the *adjacency matrix* of $\mathcal{G}$. The adjacency matrix has the property that $a_{ij} > 0$ if $(v_i, v_j) \in E$ and $a_{ij} = 0$, otherwise. The weighted degree v_i, $i \in \{1, \ldots, n\}$ is $d^{\mathrm{w}}(v_i) = \sum_{j=1}^n a_{ij}$. The *weighted degree matrix* D is the diagonal matrix defined by $(\mathsf{D})_{ii} = d^{\mathrm{w}}(i)$, for all $i \in \{1, \ldots, n\}$. The *Laplacian* is $\mathsf{L} = \mathsf{D} - \mathsf{A}$. For an undirected graph, $\mathsf{L}\mathbf{1}_n = \mathbf{1}_n^T \mathsf{L} = 0$, and $\mathsf{L} = \mathsf{L}^T$ is positive semidefinite [31]. When $\mathcal{G}$ is connected, the zero eigenvalue is simple.

D. Game Theory

We recall the class of concave games in the absence of shared constraints, see [26], [32]. A *concave game* (with unshared constraints) is a triplet $\mathbf{G} = (V, S, \{f_i\}_{i=1}^n)$, where

- V is a group of $n \in \mathbb{Z}_{\geq 1}$ players,
- $S = S_1 \times S_2 \times \ldots \times S_n$ is the strategy set, where $S_i \subset \mathbb{R}^{d_i}$, $d_i \in \mathbb{Z}_{\geq 1}$ is nonempty, convex and compact, and

- $f_i : S \to \mathbb{R}$ is the payoff for player $i \in \{1, \ldots, n\}$, a locally Lipschitz concave mapping.

A point $x^* \in S$ is called a *Nash equilibrium* of $\mathbf{G}$ if and only if, for all $i \in V$,

$$f_i(x^*) = \max_{y_i} \{ f_i(x_1^*, \ldots, x_{i-1}^*, y_i, x_{i+1}^*, \ldots, x_n^*) \mid y_i \in S_i \}.$$

In other words, when the game is at x^*, no player can improve its payoff by unilaterally deviating from this point. A celebrated theorem by Rosen guarantees the existence of Nash equilibrium for this class of games [26]. A uniqueness result can also be obtained under the so-called diagonally strict concavity assumption, along with differentiability (see [26, Theorem 4]), when one considers another suitable notion of equilibrium (the so-called normalized or variational equilibrium), see [33]. When the constraints are not shared, as it is the case in this paper, these notions of equilibria match, yielding an applicable uniqueness result.

In many applications, including the one in this paper, the differentiability assumption does not hold. Furthermore, the convergence proof of the gradient flow procedure for seeking this Nash equilibrium [26, Theorem 7] is no longer valid; however, the results are still valid when these functions are locally Lipschitz, see [34].

III. Problem Setup

In our setting, we consider a set of aggregators, each of which participates in a real-time energy market by submitting an offer to sell (or a bid to buy) certain amount of energy over some period of time of fix duration, at some price. In order to determine the amount of energy to be bought or sold, each aggregator will send requests to a group of DERs it is associated with in order to provide (or consume) energy, and will incentivize them to do so via some pricing strategy, i.e., the aggregator will offer to buy (or sell) energy via some pricing strategy. In the remainder of this section, we first introduce a model that describes the energy consumption/provision decision-making process of each DER; then, we formalize the collective DER decision-making process. Since we will not be introducing any interaction mechanism between the aggregators, and since each DER group is associated with one and only one aggregator, with no sharing of information across different groups, we can essentially concentrate, without any loss of generality, on a single aggregator and a single DER group associated with it.

A. DER model

Within a single aggregator model, after receiving a request from the aggregator, each DER will make a decision on whether or not it will provide (or consume) energy, and if it decides to do so, it will decide on the amount of energy it will provide or consume. The decision that each DER is faced with depends, among other things, on its own utility function, along with the pricing strategy designed by the aggregator. Specifically, in this paper, the final price according to which the DERs will pay when consuming energy (or be paid for when producing energy) will be a function of the average (or total) amount of energy that the DERs collectively decide to consume (or produce) in response to the aggregator's pricing strategy. Next, we formalize this setting.

Let $V = \{v_1, \ldots, v_n\}$, $n \in \mathbb{Z}_{\geq 1}$, be the set of DERs to be coordinated by a particular aggregator, and denote by $x_i(\tau) \in [0, 1]$ the energy level of each v_i, $i \in \{1, \ldots, n\}$, at time $\tau \in \mathbb{R}_{\geq 0}$. Without loss of generality, we assume that, at any point in time τ_0, each DER in V is willing to modify its energy energy level $x_i(\tau_0) = x_i^0$ to a new value $x_i^* \in [0, 1]$. We assume that the change in energy level from x_i^0 to x_i^* is not instantaneous and occurs at a rate $p_i(t)$—the power consumed or produced by the DER—satisfying $x_i^* = x_i^0 + \int_0^T p_i(t) dt$, for some appropriate T.

In order for a DER to decide whether or not it will increase or decrease its energy level, it will take into account the price that the aggregator will offer for consuming (in which case, the DER pays the aggregator and its energy level will increase), or producing (in which case, the aggregator pays the DER and its energy level will decrease). Let the function $P_{\text{c}} : [0, 1] \to \mathbb{R}_{\geq 0}$, which takes values $P_{\text{c}}(\overline{x}(\tau))$, $\overline{x}(\tau) = \frac{1}{n} \sum_{i=1}^n x_i(\tau)$, denote the price per unit of energy according to which each DER pays the aggregator when consuming energy; and let the function $P_{\text{p}} : [0, 1] \to \mathbb{R}_{\geq 0}$, which takes values $P_{\text{p}}(\overline{x}(\tau))$, denote the price per unit of energy according to which the aggregator will pay each DER when producing energy; these two functions are set in advance by the aggregator via some pricing strategy.

Additionally, we assume that in making its decision, each DER takes into account the utility of maintaining a certain level of energy. For example, in the context of PEVs, having more energy available would mean that the owner can operate the PEV longer in case it is needed. To this end, we define a set of utility functions $U_i : [0, 1] \to \mathbb{R}_{\geq 0}$, with values $U_i(x_i)$, for all $i \in \{1, \ldots, n\}$. Note that with slight abuse of notation, with $x_i(\tau)$ as the available energy at time τ, we have used x_i here to denote a variable in $[0, 1]$. Although our treatment allows for more general classes of utility functions, it is assumed that each U_i is an increasing function of the available energy, i.e., at no cost, it is beneficial for each DER to retain as much energy as possible.

Let us next describe the decision-making process that each DER is faced with. As mentioned earlier, each DER needs to decide whether or not it will modify its energy level from some initial value $x_i^0 = x_i(\tau_0)$ to some other value x_i^* within T units of time. The DERs considered in this paper are price anticipating in the sense that each DER wishes to maximize a payoff function $f_i : X \times X_{\text{agg}} \to \mathbb{R}$, where $X = [0, 1]^n$ and $X_{\text{agg}} = \mathscr{B}^0([0, 1]) \times \mathscr{B}^0([0, 1])$, given by

$$f_i(x_i, x_{-i}, P_{\text{c}}, P_{\text{p}}) = \begin{cases} U_i(x_i) - (x_i - x_i^0) P_{\text{c}}(\overline{x}), & x_i > x_i^0, \\ U_i(x_i) - (x_i - x_i^0) P_{\text{p}}(\overline{x}), & x_i \leq x_i^0, \end{cases} \tag{2}$$

where $(x_i^0, x_{-i}^0) \in X$ denotes the initial energy levels of all DERs. When the DERs make the decision about their state x_i^* at time T, their payoffs are computed according to (2). Price anticipating refers here to the fact that the ith DER is aware of the fact that the pricing strategies are evaluated at $\overline{x}$, and not at x_i; the following remark further elaborates on this.

Remark 3.1: **(Price anticipating assumption):** Similar to other scenarios of resource allocation problems (see, e.g., [23]), the assumption that the DERs are price anticipating allows us to model more realistic scenarios, where each DER is a rational player and is aware of the fact that the prices are evaluated as a function of available resources and not solely as a function of its own energy level. Note that from the aggregator's perspective, it is reasonable to select the pricing as a function of the available energy across all the DERs. For example, if all the DERs have nearly full capacity, the aggregator would need to offer a lower price in order to incentivize the DERs for consuming more energy. However, this would also give the DERs the incentive to anticipate that the price is correlated with the total available energy. Finally, note that the proposed model includes the price taking scenarios as a subset. •

B. DER-Aggregator Decision-Making Process

Each aggregator participates in a real-time energy market cleared every T units of time. For each of these time periods, the aggregator submits an offer to sell, or submits a bid to buy, a block of energy, i.e., some amount of energy $\mathcal{X} \in \mathbb{R}_{<0}$ at some price $\pi_p \in \mathbb{R}_{\geq 0}$, to be delivered over the duration of the period, which we assume, without loss of generality, to be of length T. In the remainder of the paper, we focus on the decision-making process that determines the aggregator bid or offer for a single period; the process would repeat every T units of time if the aggregator decides to participate in every single market. Hence, T will not play any role in the rest of the paper.

In order to determine $\mathcal{X}$ in each period, the aggregator broadcasts its pricing strategies $(P_\text{c}, P_\text{p}) \in X_\text{agg}$ to a group of $n_i \in \mathbb{Z}_{\geq 1}$ DERs; we assume that each DER group is determined beforehand, and we also assume that different groups do not share members. Each DER within a group will try to maximize its own payoff function f_i as defined in (2). In this regard, for a fixed pricing strategy $(P_\text{c}, P_\text{p}) \in X_\text{agg}$, the DER decision-making process defines a game, which we refer to as the DER Game, given by

$$\mathbf{G}_\text{DERs} = (V,\ [0,1]^n,\ f_1 \times \ldots \times f_n). \tag{3}$$

We are now in a position to pose some relevant questions:

(a) *(Existence of equilibria):* given the pricing strategies of the aggregator $P_\text{c}, P_\text{p} \in \mathscr{B}^0([0,1])$, does there exist a Nash equilibrium solution, denoted by $x^* \in X$, to the DER game in (3)? If so, is the equilibrium unique?

(b) *(Distributed equilibria seeking):* if the answers to both parts of (a) are in the affirmative, can the DERs use a strategy which only relies on local information available to each DER, to seek dynamically the Nash equilibrium, after the pricing strategy is fixed?

In Section IV we provide answers to the questions in (a); whereas in Section V, we provide an answer to the question in (b).

IV. Existence and Uniqueness of Equilibrium Points

In this section, we characterize the properties of the DER game in (3). We start by stating some assumptions on the payoff functions of the players.

Assumption 4.1: **(Properties of the payoff functions):** We assume that

(i) U_i are concave, nondecreasing, and continuously differentiable, for all $i \in \{1, \ldots, n\}$,

(ii) P_c is convex, twice differentiable, and nondecreasing,

(iii) P_p is concave, twice differentiable, nondecreasing, and

(iv) $P_\text{c}(\overline{x}) > P_\text{p}(\overline{x})$, for all $\overline{x} \in [0,1]$.

Assumption (i) means that without any incentive for providing energy, or cost of consuming energy, DERs would prefer to retain as much energy as possible. The nondecreasing parts of assumptions (ii) and (iii), respectively, ensure that when the average value $\overline{x}$ is high (meaning that there is a large amount of energy available), the aggregator increases the price for consuming energy and when $\overline{x}$ is low (meaning that DERs do not have enough energy available) the aggregator increases the price for providing energy.[2] The convexity and concavity assumptions of (ii) and (iii) ensure the concavity of the payoff function of each player (see Proposition 4.2 below). For example, the price per unit of energy is convex when a DER is a consumer, which captures a similar behavior to that corresponding to the effect of diminishing returns in the case of utility functions (incremental increase in satisfaction progressively gets less as the level of utility gets higher), but in the opposite direction. As there is more demand for energy, the price per unit of energy increases more steeply. Finally, the last assumption prevents DERs from simultaneously trading energy for increasing their payoffs and ensures concavity of the pricing strategy.

Proposition 4.2: **(Properties of the DER payoff function):** Under Assumption 4.1, the payoff function of each DER, given by (2), is concave in its first argument.

Proof: Since U_i is concave, it will be sufficient to prove the result for $U_i = 0$, for all $i \in \{1, \ldots, n\}$. By Lemmata A.1 and A.2 in the Appendix, and using Assumptions (i), (ii) and (iii), the function f_i is piecewise concave, i.e., f_i is concave on $[0, x_i^0)$ and $(x_i^0, 1]$. Furthermore, the set of superderivatives (see, e.g., [28]), of f_i at x_i^0 is given by $[-P_\text{c}((x_i^0 + x_{-i}^0)/n), -P_\text{p}((x_i^0 + x_{-i}^0)/n)]$. By the second order condition of concavity for f^i on $[0, x_i^0)$, we conclude that $\frac{df_i}{dx_i}(x, x_{-i}) \geq -P_\text{p}((x_i^0 + x_{-i}^0)/n)$, for all $x \in [0, x_i^0)$. A similar argument shows that $\frac{df_i}{dx_i}(x, x_{-i}) \leq -P_\text{c}((x_i^0 + x_{-i}^0)/n)$, for $x_i \in (x_i^0, 1]$. Using this observation and the property that $P_\text{c}((x_0^i + x_{-i})/n) > P_\text{p}((x_i^0 + x_{-i}^0)/n)$, the first-order condition of concavity holds for f_i on $[0,1]$. ■

Figure 1 depicts a payoff function which satisfies the conditions of Proposition 4.2. Using this result, and in view

[2] In the context of PEV charging/discharging, these assumptions are reasonable, when the request $\mathcal{X}$ matches a realistic operating scenario, where most PEVs are willing to charge overnight and discharge during the daytime.

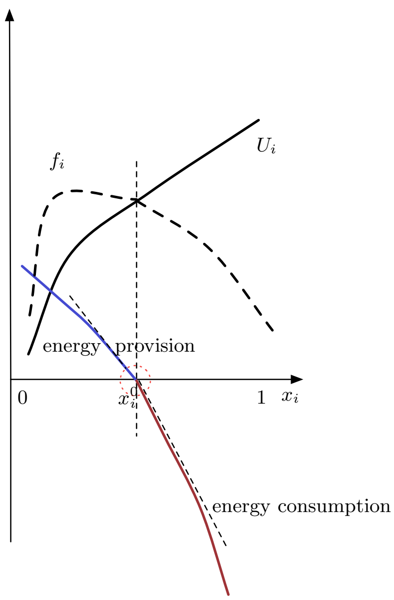


Fig. 1. Payoff function of DER v_i for a given pricing strategy and $x_{-i} \in [0,1]^{n-1}$, where x_{-i} denotes the the strategy of all players except the ith player, and fixed x_{-i}. The blue curve assigns to each point $x_i \in [0, x_i^0)$ the value of $-(x_i - x_i^0)P_{\mathrm{p}}(\overline{x})$; similarly, the red curve assigns to each point $x_i \in (x_i^0, 1]$ the values of $-(x_i - x_i^0)P_{\mathrm{c}}(\overline{x})$. In the scenario shown in the figure, the DER v_i is more likely to provide energy than to consume it.

of the fact that the strategy sets are convex, the existence of a Nash equilibrium is guaranteed for the problem at hand [26].

Theorem 4.3: **(Existence of solutions for $\mathbf{G}_{\text{DERs}}$):** Under Assumption 4.1, $\mathbf{G}_{\text{DERs}}$ has a Nash equilibrium.

An extension of [26, Theorem 4] to nonsmooth functions, see [34], can now be applied to guarantee uniqueness, under the assumption of *diagonally strict concavity*, see [26].

V. A Continuous-Time Distributed Algorithm for Seeking the Nash Equilibrium

We now design a strategy, distributed in a sense to be described shortly, which allows for the players to arrive at the Nash equilibrium, when it is unique. The strategy can be thought of as the distributed version of the gradient-flow procedure [26, Theorem 7] for seeking the Nash equilibrium, extended to include nonsmooth payoff functions. It is in continuous-time and consensus-based, and is motivated by the distributed optimization protocols in [16], [17], and the Nash-seeking strategies for noncooperative games in [35]. Although we derive our results by considering $\mathbf{G}_{\text{DERs}}$, when the strategy sets are convex and compact, and the constraints are not shared, they are readily extendable to include other concave games with unique Nash equilibrium.

Each DER can only communicate with its neighboring DERs; the exchange of information among all DERs is described by a connected undirected graph, denoted by $\mathcal{G}_{\text{DERs}}$. Each player has access to only its own payoff function, which is assumed to be concave in the corresponding argument, but not necessarily differentiable. To ensure uniqueness, we additionally assume, as a sufficient condition, that the diagonally strict concavity condition of [34] holds; as we will see later, this uniqueness assumption allows for global convergence of our Nash-seeking dynamics. We denote this unique Nash equilibrium by $x^* \in X$, $X = [0,1]^n$. Next, we describe the networking aspect of our model. We assume that each player forms an estimate of what this Nash equilibrium should be; we denote the estimate of v_i by $x^i \in \mathbb{R}^n$. We let $\boldsymbol{x}^T = (x^1, \dots, x^n) \in X^n$.

Consider now the set-valued vector field $\Psi : \mathbb{R}^{n^2} \times \mathbb{R}^{n^2} \rightrightarrows \mathbb{R}^{n^2} \times \mathbb{R}^{n^2}$ given by

$$\begin{aligned}\Psi(\boldsymbol{x}, \boldsymbol{z}) = \{(-\mathbf{L}\boldsymbol{x} - \mathbf{L}\boldsymbol{z} + \boldsymbol{s}_{\boldsymbol{x}}, \mathbf{L}\boldsymbol{x}) \mid \boldsymbol{s}_{\boldsymbol{x}} \in \mathcal{D}_{\boldsymbol{x}} := \{u \in \mathbb{R}^{n^2} \mid \\ u = \underbrace{(\eta_1, 0, \dots, 0}_{\text{computed by } v_1}, 0, \eta_2, 0, \dots, 0, 0, \dots, 0, \eta_n)^T, \\ \eta_i \in \partial_{x_i} f_i(x^i), i \in \{1, \dots, n\}\}\},\end{aligned}$$

where $\mathbf{L} = \mathsf{L} \otimes \mathsf{I}_n \in \mathbb{R}^{n^2 \times n^2}$ and L is the Laplacian of $\mathcal{G}_{\text{DERs}}$. Note that v_i evaluates $\partial_{x_i} f_i(\cdot)$ using its own estimate x^i. Here, the term computed by v_1 refers to the fact that agent v_1 can compute this value using its own payoff function. By Lemma 2.1, the set-valued mapping Ψ is nonempty and upper semicontinuous. Let $\mathcal{C} = (X^n \times \mathbb{R}^{n^2})$ and consider the dynamics $\Xi : \mathcal{C} \times (\mathbb{R}^{n^2} \times \mathbb{R}^{n^2}) \rightrightarrows \mathcal{C} \times (\mathbb{R}^{n^2} \times \mathbb{R}^{n^2})$ given by

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &\in \Pi_{\mathcal{C}}(-\mathbf{L}\boldsymbol{x}(t) - \mathbf{L}\boldsymbol{z}(t) + \mathcal{D}_{\boldsymbol{x}(t)}), \\ \dot{\boldsymbol{z}}(t) &= \mathbf{L}\boldsymbol{x}(t), \end{aligned} \tag{4}$$

where $\Pi_{\mathcal{C}}$ is the projection map on $\mathcal{C} \times (\mathbb{R}^{n^2} \times \mathbb{R}^{n^2})$ defined in Appendix VII-B. In what follows, we will occasionally refer to these dynamics as the *concave Nash-seeking dynamics*. By definition of the projection map and since there is no shared constraints, the updates for the estimates of v_i, the ith component of (4), can be computed in a distributed fashion. By Proposition A.3 and the observation above about Ψ, the solutions to (4), in the sense of Caratheorory, exist from any initial condition in $\mathcal{C}$. Next, we clarify why we are interested in studying (4) by characterizing its set of equilibria.

Lemma 5.1: **(Equilibria of** (4)**):** When $\mathcal{G}_{\text{DERs}}$ is undirected and connected, (4) has at least one equilibrium. Moreover, $(\boldsymbol{x}^*, \boldsymbol{z}^*)$ is an equilibrium if and only if $\boldsymbol{x}^* = \mathbf{1}_n \otimes x^*$, where $x^* \in X$ is Nash equilibrium of $\mathbf{G}_{\text{DERs}}$.

Proof: Using the definition of the mapping $\Pi_{\mathcal{C}}$, we have that $(\boldsymbol{x}^*, \boldsymbol{z}^*)$ is an equilibrium of Ξ if and only if

$$\mathbf{L}\boldsymbol{x}^* = \mathbf{0} \qquad \text{and} \qquad -\mathbf{L}\boldsymbol{z}^* + \boldsymbol{s}_{\boldsymbol{x}} = \boldsymbol{\beta},$$

where $\boldsymbol{\beta} \in \ker(\Pi_{\mathcal{C}})$. Since the network is connected, $\boldsymbol{x}^* = \mathbf{1}_n \otimes x^*$, for some $x^* \in X^n$. From the definition of the mapping $\Pi_{\mathcal{C}}$, we then conclude that

$$-\mathbf{L}\boldsymbol{z}^* + \boldsymbol{s}_{\boldsymbol{x}^*} = \boldsymbol{\beta}. \tag{5}$$

Note that

$$(\mathbf{1}_n^T \otimes \mathsf{I}_n)\mathbf{L} = (\mathbf{1}_n^T \otimes \mathsf{I}_n)(\mathsf{L} \otimes \mathsf{I}_n) = \mathbf{1}_n^T \mathsf{L} \otimes \mathsf{I}_n = \mathbf{0}_{n \times n^2},$$

where the last equality follows since the network is undirected. Using (5), we have $(\mathbf{1}_n^T \otimes \mathsf{I}_n)\boldsymbol{s}_{\boldsymbol{x}^*} = (\mathbf{1}_n^T \otimes \mathsf{I}_n)\boldsymbol{\beta}$. Let $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$, where $\beta_i \in \mathbb{R}^n$. Further note that, by definition, the jth component of β_i, denoted by $(\beta_i)_j$ is zero, for all $i \in \{1, \dots, n\}$, if and only if $0 < x_j^* < 1$. Moreover, this component is negative or positive, if, respectively $x_j^* = 0$ or $x_j^* = 1$. Thus, using also Lemma 2.2, we conclude that $\eta_j - \gamma_j = 0$ and $\eta_j + \gamma_j' = 0$, where $\gamma_j = |\sum_{i=1}^n (\beta_i)_j|$ when $x_j^* = 0$ and is zero otherwise, and $\gamma_j' = |\sum_{i=1}^n (\beta_i)_j|$ when $x_j^* = 1$ and is zero otherwise, i.e.,

$$0 \in \partial_{x_j} f_j((x^*)^j) + \gamma_j \partial_{x_j}(1 - x_j) + \gamma_j' \partial_{x_j}(x_j),$$

for all $j \in \{1, \ldots, n\}$. Thus x^* satisfies the Kuhn-Tucker necessary conditions and, since the constraints are not shared, by [26, Theorem 3], it is the unique Nash equilibrium of $\mathbf{G}_{\text{DERs}}$. ■

Our goal in the balance of this section is to characterize the convergence properties of (4).

Theorem 5.2: **(Asymptotic convergence of** (4)**):** When $\mathcal{G}_{\text{DERs}}$ is undirected and connected, the dynamics in (4) are asymptotically convergent. Moreover, the projection onto the first component of its trajectory converges to $\boldsymbol{x}^* = \mathbf{1}_n \otimes x^*$, where $x^* \in \mathbb{R}^n$ is the Nash equilibrium of $\mathbf{G}_{\text{DERs}}$.

Proof: For an initial condition $(\boldsymbol{x}^0, \boldsymbol{z}^0)$, consider the set $V^{-1}(\leq V(\boldsymbol{x}^0, \boldsymbol{z}^0)) = \{(\boldsymbol{x}, \boldsymbol{z}) \in X^n \times \mathbb{R}^n \mid 0 \leq V(\boldsymbol{x}, \boldsymbol{z}) \leq V(\boldsymbol{x}^0, \boldsymbol{z}^0)\}$. Let $(\boldsymbol{x}^*, \boldsymbol{z}^*) \in X^n \times \mathbb{R}^n$ be an equilibrium of (4) such that $\mathbf{1}^T \boldsymbol{z}^0 = \mathbf{1}^T \boldsymbol{z}^*$, and consider a candidate Lyapunov function $V : X^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$,

$$V(\boldsymbol{x}, \boldsymbol{z}) = \frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}^*)^T(\boldsymbol{x} - \boldsymbol{x}^*) + \frac{1}{2}(\boldsymbol{z} - \boldsymbol{z}^*)^T(\boldsymbol{z} - \boldsymbol{z}^*). \quad (6)$$

This function is clearly a smooth mapping. Let us examine its set-valued Lie derivative. For each $\xi \in \widetilde{\mathcal{L}}_\Xi V(\boldsymbol{x}, \boldsymbol{z})$, using the definition of the projection map, there exists $v = -(\mathbf{L}\boldsymbol{x} + \mathbf{L}\boldsymbol{z} - \boldsymbol{s}_{\boldsymbol{x}} + \gamma - \gamma') \in \Xi(\boldsymbol{x}, \boldsymbol{z})$, with $\boldsymbol{s}_{\boldsymbol{x}} \in \mathcal{D}_{\boldsymbol{x}}$, such that

$$\begin{aligned}\xi = v^T \nabla V(\boldsymbol{x}, \boldsymbol{z}) = (\boldsymbol{x} - \boldsymbol{x}^*)^T\big(-(\mathbf{L}\boldsymbol{x} + \mathbf{L}\boldsymbol{z} - \boldsymbol{s}_{\boldsymbol{x}} + \gamma - \gamma')\big) \\ + (\boldsymbol{z} - \boldsymbol{z}^*)^T(\mathbf{L}\boldsymbol{x}), \quad (7)\end{aligned}$$

where the nonzero components of γ_i and γ_i', $i \in \{1, \ldots, n^2\}$, correspond to the projections onto $\mathcal{C} \times \mathbb{R}^n \times \mathbb{R}^n$. Next, let $\Phi : X^n \times X^n \to \mathbb{R}$ be a mapping given by

$$\begin{aligned}\Phi(\boldsymbol{x}, \boldsymbol{y}) = &\sum_{i=1}^n f_i(y_1^i, \ldots, y_{i-1}^i, x_i^i, y_{i+1}^i \ldots, y_n^i) \\ &+ \gamma_i(1 - x_i^i) + \gamma_i'(x_i^i).\end{aligned}$$

If $\zeta_{(\boldsymbol{x}, \boldsymbol{y})} \in \partial_{\boldsymbol{x}} \Phi(\boldsymbol{x}, \boldsymbol{y})$, then

$$\zeta_{(\boldsymbol{x}, \boldsymbol{y})} = ((\zeta_1, 0, \ldots, 0), (0, \zeta_2, 0 \ldots, 0), \ldots, (0, 0, \ldots, \zeta_n)),$$

where $\zeta_i \in \partial_{x_i} f_i(y_1^i, \ldots, y_{i-1}^i, x_i^i, y_{i+1}^i \ldots, y_n^i) + \gamma_i \partial_{x_i}(1 - x_i^i) + \gamma_i' \partial_{x_i}(x_i^i)$. By definition, there exists $\boldsymbol{s}_{\boldsymbol{x}} \in \mathcal{D}_{\boldsymbol{x}}$ such that $\boldsymbol{s}_{\boldsymbol{x}} + \gamma - \gamma' = \zeta_{(\boldsymbol{x}, \boldsymbol{x})}$. Let us define next the mapping $F_{\boldsymbol{y}} : X^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$F_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{z}) = \frac{1}{2}\boldsymbol{x}^T \mathbf{L}\boldsymbol{x} + \boldsymbol{x}^T \mathbf{L}\boldsymbol{z} - \Phi(\boldsymbol{x}, \boldsymbol{y}).$$

Since the Laplacian matrix is positive semidefinite and f_i is concave in all its arguments, this function is convex and linear in its first and second arguments, respectively. One can additionally observe, using [26, Theorem 3], that the Nash equilibrium of $\mathbf{G}$ corresponds to the saddle points of $F_{\boldsymbol{x}}$. Recall now that $\mathbf{L}$ is symmetric. Thus, using $\mathbf{L}\boldsymbol{x} + \mathbf{L}\boldsymbol{z} - \boldsymbol{s}_{\boldsymbol{x}} + \gamma - \gamma' \in \partial_{\boldsymbol{x}} F_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{z})|_{\boldsymbol{y} = \boldsymbol{x}}$, and the first order condition of convexity, we deduce that $(\boldsymbol{x}^* - \boldsymbol{x})^T(\mathbf{L}\boldsymbol{x} + \mathbf{L}\boldsymbol{z} - \boldsymbol{s}_{\boldsymbol{x}} + \gamma - \gamma') \leq F_{\boldsymbol{x}}(\boldsymbol{x}^*, \boldsymbol{z}) - F_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{z})$. On the other hand, the linearity of $F_{\boldsymbol{x}}$ in its second argument implies that $(\boldsymbol{z} - \boldsymbol{z}^*)^T(\mathbf{L}\boldsymbol{x}) = F_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{z}) - F_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{z}^*)$. Thus we conclude

$$\begin{aligned}\xi \leq &F_{\boldsymbol{x}}(\boldsymbol{x}^*, \boldsymbol{z}) - F_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{z}) + F_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{z}) - F_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{z}^*) \\ = &F_{\boldsymbol{x}}(\boldsymbol{x}^*, \boldsymbol{z}) - F_{\boldsymbol{x}}(\boldsymbol{x}^*, \boldsymbol{z}^*) + F_{\boldsymbol{x}}(\boldsymbol{x}^*, \boldsymbol{z}^*) - F_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{z}^*).\end{aligned}$$

Since the equilibria of Ξ are the saddle points of $F_{\boldsymbol{x}}$, we deduce that $\max \widetilde{\mathcal{L}}_\Xi V(\boldsymbol{x}, \boldsymbol{z}) \leq 0$. As a by-product, the trajectories of (4) are bounded.

Finally, recall that the map Ξ is upper semicontinuous and nonempty on $\mathcal{C}$. With all the above hypotheses satisfied, the application of the LaSalle invariance principle for the set-valued dynamical system, cf. Theorem 2.4, implies that, from a given initial condition, the evolution of (4) approaches a set M of the form $V^{-1}(c) \cap S$, $c \in \mathbb{R}$, where S is the largest weakly positively invariant set contained in

$$\begin{aligned}\{(\boldsymbol{x}, \boldsymbol{z}) \in V^{-1}(\leq V(\boldsymbol{x}^0, \boldsymbol{z}^0)) \mid \text{there exists} \\ (\boldsymbol{x}', \boldsymbol{z}') \in \Xi(\boldsymbol{x}, \boldsymbol{z}) \text{ such that } V(\boldsymbol{x}', \boldsymbol{z}') = V(\boldsymbol{x}, \boldsymbol{z})\}.\end{aligned}$$

In order to complete the proof, we need to show that $c = 0$. Let us suppose otherwise. Note that when $(\boldsymbol{x}, \boldsymbol{z}) \in M$, we have that $F_{\boldsymbol{x}}(\boldsymbol{x}^*, \boldsymbol{z}^*) - F_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{z}^*) = 0$, i.e.,

$$-\Phi(\boldsymbol{x}^*, \boldsymbol{x}^*) - \frac{1}{2}\boldsymbol{x}^T \mathbf{L}\boldsymbol{x} - \boldsymbol{x}^T \mathbf{L}\boldsymbol{z}^* + \Phi(\boldsymbol{x}, \boldsymbol{x}) = 0. \quad (8)$$

Define now the function $G : X^n \times \mathbb{R}^n \to \mathbb{R}$ given by $G_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{z}) = -\Phi(\boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{z}^T \mathbf{L}\boldsymbol{x}$. Note that this function is convex in its first argument and linear in its second. Moreover, $G_{\boldsymbol{x}}$ has the same saddle points as $F_{\boldsymbol{x}}$. As a result, $G_{\boldsymbol{x}}(\boldsymbol{x}^*, \boldsymbol{z}^*) - G_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{z}) \leq 0$. This observation, along with (8), yields that $\mathbf{L}\boldsymbol{x} = 0$, when $(\boldsymbol{x}, \boldsymbol{z}) \in M$, and thus we have $-\Phi(\boldsymbol{x}^*, \boldsymbol{x}^*) + \Phi(\boldsymbol{x}, \boldsymbol{x}) = 0$. By uniqueness and the definition of Φ, we thus conclude that $\boldsymbol{x} = \boldsymbol{x}^*$. ■

It is worth mentioning that (4) allows for each DER to arrive at the Nash equilibrium in a distributed way, without having access to the private information of others, in particular, other DERs' utility functions, provided that they all follow the same rules. In the context of DERs, the mechanism described above can be offered as a built-in system by the aggregator, with the functionality of personalized utility settings, so that the DERs can automatically decide on the amount of consumption and provision of energy over a certain period of time. We conclude this section with two remarks on unidirectional communication structures, and discrete-time implementation of the pricing inputs.

Remark 5.3: **(Directed network topology):** Our proof of Theorem 5.2 heavily relies on the symmetry of the Laplacian matrix, a result of the fact that the network is undirected. In fact, one can construct an example of a weight-balanced strongly connected network for which these dynamics are divergent. Additionally, given that the payoff function of each agent is inherently only locally Liptchitz, the proof techniques used in [17] are not applicable; nevertheless, similar algorithms may still be convergent. •

Remark 5.4: **(Implementation of the continuous-time controller):** Although we have designed the pricing inputs as

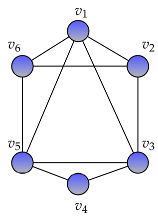


Fig. 2. Network of DERs for the case study is shown. The adjacency matrix associated with this network has 1s in the entries corresponding to edges and 0s elsewhere.

a continuous-time controller, it can potentially be implemented in a discrete-time manner; this is a reasonable assumption as the change in pricing happens much slower than the Nash-seeking dynamics. Finally, it is worth mentioning that the procedure proposed is robust, following form a property of the saddle-point dynamical systems, see [36], and hence can regulate slow time-varying requests. An analytic characterization of the discretization time step is a subject for future work. •

VI. Numerical Simulations

Consider a group of DERs associated with an aggregator; for illustration purposes, we have only selected six DERs $\{v_1, \ldots, v_6\}$. The DERs are price anticipating and can obtain information from each other via a communication network shown in Figure 2. Note that this network is connected.

Each DER utility function $U_i : [0, 1] \to \mathbb{R}_{\geq 0}$, $i \in \{1, \ldots, 6\}$, is given by

$$U_i(x) = u_i^1 \log(1 + x) + u_i^2 x,$$

where U_i is normalized so that $u_i^1, u_i^2 \in (0, 1]$. Note that U_i is increasing and strictly concave, and thus satisfies Assumption 4.1(i). Let us assume that the aggregator has linear pricing strategies given by

$$P_{\mathrm{c}}(\overline{x}) = c_1\overline{x} + c_2, \qquad \text{and} \qquad P_{\mathrm{p}}(\overline{x}) = d_1\overline{x} + d_2,$$

where these functions are normalized so that $c_1, d_1 \in (0, 1]$ and $c_2, d_2 \in [0, 1]$. The payoff functions of each DER v_i, $i \in \{1, \ldots, 6\}$, is given by (2). Note that these pricing functions satisfy Assumption 4.1(ii-iii). Also, by assuming that $c_1 > d_1$ and $c_2 > d_2$, Assumption 4.1(iv) holds true and as a result, the payoff function of each DER is concave (see Proposition 4.2). In fact, one can easily verify that each of these functions additionally satisfies the diagonally strict concavity assumption, and the Nash equilibrium of the $\mathbf{G}_{\text{DERs}}$ with these pricing strategies is unique. In what follows, we study the evolutions of DER energy provision/consumption processes for different pricing strategies offered by the aggregator.

A. High Price for Consumption/Provision is Encouraged

We start our set of case studies with a scenario in which the aggregator wants to submit an offer; thus, it needs to encourage the DERs to provide energy. For this reason, the aggregator chooses the pricing parameters as $c_1 = 0.9$, $c_2 = 0.9$, $d_1 = 0.8$, and $d_2 = 0.8$. The parameters associated with each DER are given in Table I. We consider two cases.

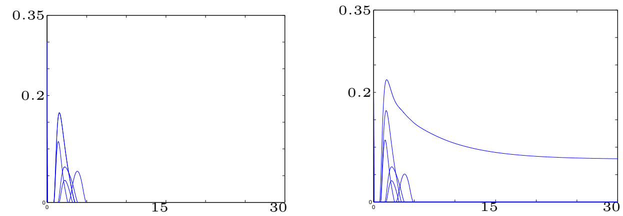
- **Case-1:** all DERs have low initial available energy and no incentive for consuming energy;
- **Case-2:** all DERs have low initial energy available; the only DER with incentive for consuming energy is v_5.

Figure 4 shows the evolution of (4) for each DER in these two cases; the value of the Nash equilibrium for each case, which all the DERs have agreed on, is given in Table I. Unlike **Case-1**, in **Case-2** v_5, has a higher incentive to absorb energy (see the value of u_5 in Table I).

B. High Price for High Consumption/Provision is not Encouraged

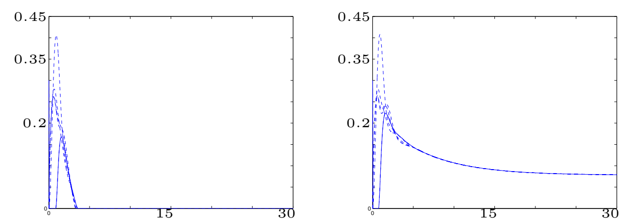
Next, we consider a scenario in which the aggregator increases the price for consuming energy when the average energy available is high. However, when the average value is low, the price paid for consuming energy is not high. We additionally assume that the aggregator does not encourage the provision of energy. The DERs' parameters are given as in Table I. We again consider two cases:

- **Case-3:** all DERs have low initial energy available and no incentive for providing energy;



(a) (b)

Fig. 3. (a) and (b) show the evolutions of (4) with respect to time for each DER in **Case-1** and **Case-2**, respectively. The horizontal and vertical axes, respectively, show the time and the available energy within each DER at each time. The initial energy profile within the DERs is given by $x_0 = (0.1, 0.2, 0.1, 0.2, 0.3, 0.2)^T$. In both scenarios, all DERs have decided to provide energy for the grid, however, in (b) DER v_5 does not have the incentive to provide all its available resources, due to its high utility.



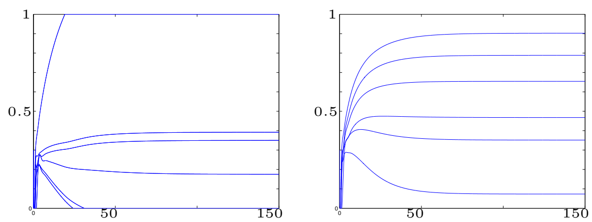
(a) (b)

Fig. 4. (a) and (b) show the consensus between the groups of DERs on the amount of energy that DER v_5 is willing to absorb/consume in the next time step. The horizontal and vertical axes, respectively, show the time and the energy level. The dashed lines show the other DERs' estimates of state value of v_5, and the solid line is the v_5's estimate on its own state.

		DER 1	DER 2	DER 3	DER 4	DER 5	DER 6
Case-1	u^1	0.1602	0.1126	0.2502	0.2113	0.3216	0.1984
	u^2	0.1250	0.1457	0.1780	0.1078	0.1162	0.1942
	x^*	0	0	0	0	0	0
Case-2	u^1	0.1602	0.1126	0.2502	0.2113	*0.8216*	0.1984
	u^2	0.1250	0.1457	0.1780	0.1078	0.1162	0.1942
	x^*	0	0	0	0	0.0780	0
Case-3	u^1	0.1602	0.1126	0.2502	0.2502	0.8216	0.1984
	u^2	0.1250	0.1457	0.1780	0.1078	0.1162	0.1942
	x^*	0	0	0.3920	0.1751	1.000	0.3499
Case-4	u^1	0.4021	0.7126	0.5502	0.6002	0.8216	0.7984
	u^2	0.1	0.1	0.1	0.1	0.1162	0.1
	x^*	0.0733	0.6543	0.3516	0.4676	0.9025	0.7885

TABLE I

THE PROPERTIES OF EACH DER ARE GIVEN. THE PRICING PARAMETERS ARE GIVEN BY $c_1 = 0.9, c_2 = 0.9, d_1 = 0.8, d_2 = 0.8$ FOR **CASE-1** AND **CASE-2** AND $c_1 = 0.7, c_2 = 0.1, d_1 = 0.1, d_2 = 0.1$ FOR **CASE-3** AND **CASE-4**. $x_0 = (0.1, 0.2, 0.1, 0.2, 0.3, 0.2)^T$.



(a) (b)

Fig. 5. (a) and (b) show the evolutions of (4) with respect to time for each DER in **Case-3** and **Case-4**, respectively. The horizontal and vertical axes, respectively, show the time and the available energy for each DER at each time. The initial energy profile within the DERs is given by $x_0 = (0.1, 0.2, 0.1, 0.2, 0.3, 0.2)^T$. In both scenarios, all DERs have decided to consume energy, however, in (a) DER v_5 has the incentive to reach its full capacity, due to its high utility.

- **Case-4:** all DERs have low initial energy available; the only DER with incentive for consuming energy is v_5.

In **Case-3**, the average initial energy available in the DERs is low and thus all DERs start by consuming energy. In particular, v_5 (with higher incentive, see u_5 in **Case-3** of Table II) can reach its full capacity quickly; however, as the average increases, the rest of the DERs are discouraged from consuming energy. In **Case-4**, we have considered a scenario in which most of the DERs have high demands; see **Case-4** of Table II. As a result, there is a higher competition for higher consumption between DERs and thus, with the exception of v_1, all the DERs manage to consume energy.

VII. CONCLUDING REMARKS

We have introduced a framework for the aggregated response for energy provision/consumption processes in networked DERs via pricing strategies. In this framework, a group of aggregators is responsible for providing a certain amount of energy over a period of time, predetermined by some market-clearing mechanism. The DERs are assumed to be price anticipating and also have their own individual utility functions; they can also collect information, the average energy available for most parts, from their neighboring DERs.

We have formulated the overall scenario as a multi-layer game, with a single aggregator as focus, in which the aggregator is a leader and sets the pricing strategies for consumption of energy and providing it. The DERs are the followers, and after receiving the pricing strategies, evaluate their next consumption/provision levels. Given the pricing strategy, we provide further the conditions under which this game is a concave game and determine in a distributed way conditions under which it has a unique Nash equilibrium. Finally, we introduce a continuous-time set-valued dynamical system, distributed over the network of DERs, which allows the DERs to compute the Nash equilibrium, when unique. We have proved that this dynamical system is asymptotically convergent, when the network of DERs is connected and undirected.

The most important future direction for research that emerges from this paper is the characterization of optimal pricing strategies for the aggregator, in the context of mechanism design. Future work will also focus on the analytic characterization of the discretization time step, extension of the convergence results to communication networks described by directed graphs, studying groups of aggregators and their interactions, and robustness and resilience in pricing strategies.

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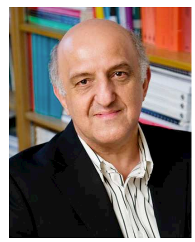

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APPENDIX

This appendix contains two auxiliary results, used in the proof of Proposition 4.2, along with some technical details about projection mappings, in particular, on set-valued vector fields.

A. Auxiliary results

This section consists of two results that are used in our proofs.

Lemma A.1: Let $f : U \to \mathbb{R}_{\geq 0}$, $U \subset \mathbb{R}$, be convex, nondecreasing, and twice differentiable. Then the mapping $g : \mathbb{R} \to \mathbb{R}$, given by $g(x) = -(x - x^0)f(x)$, where $x^0 \in \mathbb{R}$, is nonincreasing and concave on $S = \{x \in \mathbb{R} \mid x > x^0\}$.

Proof: Since f is twice differentiable, we have $\frac{dg}{dx}(x) = -f(x) - (x - x^0)\frac{df}{dx}(x)$, which is nonpositive, since f has nonnegative image and $\frac{df}{dx}(x) \geq 0$, for all $x \in \mathbb{R}$. Moreover, $\frac{d^2g}{dx^2}(x) = -2\frac{df}{dx}(x) - (x - x^0)\frac{d^2f}{dx^2}(x)$. By the second order condition of convexity, $\frac{d^2f}{dx^2}(x) \geq 0$ and since $\frac{df}{dx}(x) \geq 0$, the result follows. ∎

Lemma A.2: Let $f : U \to \mathbb{R}$, $U \subset \mathbb{R}$, be concave, nondecreasing, and twice differentiable. Then the mapping $g : \mathbb{R} \to \mathbb{R}$, given by $g(x) = -(x - x^0)f(x)$, where $x^0 \in \mathbb{R}$, is concave on $S = \{x \in \mathbb{R} \mid x < x^0\}$.

Proof: Since f is twice differentiable, we have that $\frac{d^2g}{dx^2}(x) = -2\frac{df}{dx}(x) - (x - x^0)\frac{d^2f}{dx^2}(x)$. By the second order condition of convexity, $\frac{d^2f}{dx^2}(x) \leq 0$ and since $\frac{df}{dx}(x) \geq 0$, the result follows. ∎

B. Projection maps

We recall some properties of projection maps. Consider the dynamical system

$$\dot{x}(t) = X(x(t)), \tag{9}$$

where $x(t) \in \mathbb{R}^d$, for all $t \in \mathbb{R}_{\geq 0}$, and $X : \mathbb{R}^d \to \mathbb{R}^d$. We assume that the mapping X is locally Liptchitz, yielding that a solution to this dynamical system, in the classical sense, exist [30]. Let $\mathcal{C}$ be a convex set defined by $\mathcal{C} = \{x \in \mathbb{R}^d \mid c(x) \leq 0\}$, where $c : \mathbb{R}^d \to \mathbb{R}$ is a differentiable convex function (in fact, what we establish next can easily be extended to the case where c is only locally Liptchitz convex). Let us also suppose that $\mathcal{C}$ is compact and nonempty in $\mathbb{R}^d$ and denote its boundary set by $\partial\mathcal{C} = \mathcal{C}/\mathring{\mathcal{C}}$.

We define the *projection map* $\pi_{\mathcal{C}} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ on $\mathcal{C}$ by

$$\pi_{\mathcal{C}}(X(x)) = \begin{cases} X(x) & x \in \mathring{\mathcal{C}} \text{ or } x \in \partial\mathcal{C} \\ & \quad \text{and } \langle \nabla_x c(x), X(x) \rangle \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that this mapping is not continuous on $\mathcal{C} \times \mathbb{R}^d$. Nevertheless, a solution to this dynamical system, in the Caratheorory sense, exists. This is because this vector field is directionally continuous, see [37]. An alternative practical approach that is common in the literature, see for example [38], is to approximate the vector field close to the boundary set, as we recall next. Let $\epsilon \in \mathbb{R}_{>0}$ and let $\mathcal{C}_\epsilon := \{x \in \mathbb{R}^d \mid c(x) \leq \epsilon\}$. We define the mapping $\tilde{\pi}_{\mathcal{C}} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ by

$$\tilde{\pi}_{\mathcal{C}}(X(x)) = \begin{cases} X(x) & x \in \mathring{\mathcal{C}} \\ & \quad \text{or } x \in \partial\mathcal{C} \text{ and } \langle \nabla_x c(x), X(x) \rangle \leq 0 \\ (\mathsf{I}_d - \alpha(x)\frac{\nabla_x c(x)\nabla_x^T c(x)}{\nabla_x^T c(x) M \nabla_x c(x)})X(x) \\ & \quad x \in \mathcal{C}_\epsilon/\mathring{\mathcal{C}} \text{ and} \langle \nabla_x c(x), X(x) \rangle \geq 0, \end{cases} \tag{10}$$

where $\alpha(x) = \min\{1, \frac{c(x)}{\epsilon}\}$. In [38], it has been shown that the mapping $\tilde{\pi}_{\mathcal{C}}$ defined above is locally Liptchitz; moreover more importantly, the evolution of the dynamical system

$$\dot{x}(t) = \tilde{\pi}_{\mathcal{C}}(X(x))$$

is bounded in $\mathcal{C}_\epsilon$. For the purpose of our work, we present an extension of the projection map which acts on set-valued dynamical systems. We define the projection of the differential inclusion (1) by

$$\dot{x}(t) \in \Pi_{\mathcal{C}}(\Psi(x(t))) \tag{11}$$

where $t \in \mathbb{R}_{\geq 0}$ and the mapping $\Pi_{\mathcal{C}}$, evaluated at $x \in \mathbb{R}^d$, maps the set of subsets of $\mathbb{R}^d$ to itself, i.e., the mapping $\Pi_{\mathcal{C}}|_x : 2^{\mathbb{R}^d} \to 2^{\mathbb{R}^d}$, and is defined by

$$\Pi_{\mathcal{C}}(\Psi(x)) = \begin{cases} \Psi(x) & x \in \mathring{\mathcal{C}} \text{ or } x \in \partial\mathcal{C} \\ & \quad \text{and } \langle \nabla_x c(x), X(x) \rangle \leq 0, \\ & \quad \text{for all } X(x) \in \Psi(x) \\ \mathscr{M}(\{0\}, \{X(x) \in \Psi(x) \mid \langle \nabla_x c(x), X(x) \rangle \leq 0\}) \\ & \quad \text{otherwise,} \end{cases} \tag{12}$$

where $\mathscr{M}(A, B) = A$ if B is the empty set, and is A otherwise. Note that, by construction, if a solution to (11) exists from an initial condition in $\mathring{\mathcal{C}}$, then this solution will always stay in $\mathcal{C}$. By definition of $\Pi_{\mathcal{C}} \circ \Psi$ and the assumption, the existence of solutions from any point in $\mathcal{C}$ can be guaranteed using Lemma 2.3.

Proposition A.3: **(Existence of solutions to (11)):** Let the set-valued mapping Ψ be locally bounded and upper semi-continuous, and take nonempty, compact, and convex values. Then, solutions from any initial condition in $\mathcal{C}$ exist.

One can also construct extensions of the smooth projection mapping of (10) to set-valued mappings; such an extension is used for the purpose of simulations in Section VI.