Time-Optimal Control in Dc-Dc Converters: A Maximum Principle Perspective

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***Abstract*—In this paper, we present an analytical framework to investigate time-optimal trajectories in dc-dc buck, boost, and buck-boost converters using Pontryagin's maximum principle. Particularly, we evaluate cases where a time-optimal transfer of converter states can be achieved with a single switching action, and furthermore, determine if such a control can be uniquely synthesized.**

I. Introduction

Applications such as high-performance processors require dc-dc power supplies that react quickly to large (and almost instantaneous) load changes in the minimum possible time. In such settings, traditional feedback control methods might offer unsatisfactory performance; thus, necessitating the development of advanced nonlinear control methods [1], [2]. For instance, time-optimal and proximate time-optimal control methods have been widely applied to improve transient performance of dc-dc converters. These methods include (but are not limited to): approximating the ideal switching curve for the desired operating point [3]–[5], pre-calculating and storing the ideal switching surface [6], [7], and utilizing charge balancing techniques [8]–[10].

It is widely accepted that time-optimal trajectories in dc-dc converters involve at most one switching action, i.e., a single change in the switch state. As justification, many works cite [11], [12], in which control laws that guarantee transfer of converter states with one switching action were developed by investigating state trajectories. However, the authors in [11], [12] do not prove that single-switch trajectories correspond to a minimum-time transfer of converter states. Single-switch start-up was shown through numerical simulations to be time optimal for representative buck and boost converters in [6]. In [7], the authors suggest that LaSalle's seminal work on linear systems in [13] establishes single-switch time-optimal control of buck converters and inverters; however, this has not been explored in detail.

To the best of our knowledge, a rigorous framework to study time-optimal trajectories in dc-dc converters has been lacking. The exception is the work in [14], where the authors use the maximum principle to study time-optimal trajectories in a buck converter. In this paper, we significantly extend the effort in [14] by developing a general framework to explore: i) whether optimal controls can be uniquely synthesized (irrespective of initial and final conditions), and ii) whether time-optimal trajectories involve at most one change in the switch state.

The remainder of this paper is organized as follows. In Section II, we introduce the maximum principle. In Section III, we show that the dynamics of the dc-dc buck, boost, and buck-boost converters can be cast as a bilinear system, and explore the existence of time-optimal controls for this class of systems. Time-optimal trajectories for the dc-dc buck converter are analyzed in Section IV. The equivalent problems for the boost and buck-boost converters are presented in Section V. Concluding remarks, open questions, and directions for future work are provided in Section VI.

II. Statement of the Maximum Principle

In this section, we present a brief statement of the maximum principle. We will then apply the maximum principle to investigate time-optimal trajectories in dc-dc converters.

Consider the control system described by

$$\dot{x} = f(x, u), \quad x(t_0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathcal{U} \subset \mathbb{R}$ is the control, t_0 is the initial time, and $x_0 \in \mathbb{R}^n$ is the initial condition. The cost function to be minimized is of the form

$$J(u) = \int_{t_0}^{t_f} L(x(t), u(t))dt, \tag{2}$$

where t_f denotes the terminal time, and $L : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ denotes the running cost. Let $x^* : [t_{0,} t_f] \to \mathbb{R}^n$ be the optimal state trajectory corresponding to the globally optimal control $u^* : [t_0, t_f] \to \mathcal{U}$. Then, there exist a function $p^* : [t_0, t_f] \to \mathbb{R}^n$ and a constant p_0^* satisfying $[p_0^* \; p^*(t)]^{\mathrm{T}} \neq 0 \; \forall \, t \in [t_0 \, t_f]$, and $p_0^* \leq 0$, such that:

1) x^* and p^* satisfy the canonical equations given by

$$\frac{dx^*}{dt} = \frac{\partial}{\partial p} H(x^*, u^*, p^*, p_0^*), \tag{3}$$

$$\frac{dp^*}{dt} = -\frac{\partial}{\partial x} H(x^*, u^*, p^*, p_0^*), \tag{4}$$

with boundary conditions $x^*(t_0) = x_0$, and $x^*(t_f) = x_f$, and where the *Hamiltonian*, H, is defined as

$$H(x, u, p, p_0) := p_0^{\mathrm{T}} L(x, u) + p^{\mathrm{T}} f(x, u). \tag{5}$$

2) The Hamiltonian has a global maximum at $u = u^*(t)$, $\forall\, t \in [t_0,\, t_f]$

$$H(x^*,\, u^*,\, p^*,\, p_0^*) \geq H(x^*,\, u,\, p^*,\, p_0^*). \quad (6)$$

3) The Hamiltonian is zero along the optimal trajectory

$$H(x^*(t),\, u^*(t),\, p^*(t),\, p_0^*) = 0 \ \ \forall\, t \in [t_0,\, t_f]. \quad (7)$$

The statement of the maximum principle above is adopted from [15]. For a detailed proof of the maximum principle, refer to [15], [16]. In the forthcoming discussion, we drop the superscript $(\cdot)^*$ when it is implicitly understood that we refer to optimal trajectories.

III. Dc-Dc Converter Dynamics

The dynamics of the buck, boost, and buck-boost converters can be expressed in state-space form as follows:

$$\dot{x} = \left(A_1 u + A_0(1-u)\right) x + \left(B_1 u + B_0(1-u)\right), \quad (8)$$

where the inductor current, i_L, and capacitor voltage, v_C, are adopted as the system states, i.e., $x = [i_L\ v_C]^{\mathrm{T}}$, and the switch state (i.e., the control input) $u \in \mathcal{U} = \{0,\, 1\}$. The system state-vector evolution is governed by $(A_1,\, B_1)$ when $u = 1$, and by $(A_0,\, B_0)$ when $u = 0$. The entries of A_0, A_1, B_0, B_1 are a function of the inductance, L, capacitance, C, and load resistance, R, as shown in Table I.

The problem of interest is to determine the number of times u switches value for a time-optimal transfer of the system state from x_0 to x_f. Based on (2), we see that for time-optimal control problems, the cost function, $L(x,u) \equiv 1$. The Hamiltonian for the above problem is therefore given by

$$\begin{aligned} H &= p_0 + p^{\mathrm{T}}\left(\left(A_1 u + A_0(1-u)\right) x + \left(B_1 u + B_0(1-u)\right)\right) \\ &= p_0 + p^{\mathrm{T}}\left(\left(A_1 - A_0\right) x + \left(B_1 - B_0\right)\right) u + p^{\mathrm{T}}\left(A_0 x + B_0\right). \end{aligned} \quad (9)$$

From the second equality in (9), it is clear that the maximization of the Hamiltonian requires u to be chosen based on the sign of the *switching function*

$$\phi = p^{\mathrm{T}}\left(\left(A_1 - A_0\right) x + \left(B_1 - B_0\right)\right), \quad (10)$$

as follows:

$$u = 1 \quad \text{if} \quad \phi > 0, \quad u = 0 \quad \text{if} \quad \phi < 0. \quad (11)$$

The value of ϕ depends on the costate, p. From (4), we see that the evolution of the costate is governed by

$$\dot{p} = \frac{\partial}{\partial x} H = -\left(A_1^{\mathrm{T}} u + A_0^{\mathrm{T}}(1-u)\right) p. \quad (12)$$

The control cannot be uniquely determined if $\phi \equiv 0$ over some time interval—the corresponding trajectories are called *singular trajectories*. To investigate this possibility, we will determine if the derivative of ϕ, which we denote by $\dot{\phi}$, can also vanish. For the system in (8), we get

$$\dot{\phi} = p^{\mathrm{T}}\left(\left(A_1 A_0 - A_0 A_1\right) x + \left(A_1 B_0 - A_0 B_1\right)\right). \quad (13)$$

For time-optimal problems, $p(t) \neq 0, \forall t > 0$. This is to enforce the non-triviality condition in the maximum principle $[p_0,\, p(t)]^{\mathrm{T}} \neq 0, \forall t > 0$, while ensuring that the Hamiltonian, $H = 0$, along the optimal trajectory. Thus, from the definition of ϕ and $\dot{\phi}$ in (10), (13), we see that for the buck, boost, and buck-boost converter topologies in Table I, singular trajectories can be ruled out if the vectors

$$\rho_1 = \left(A_1 - A_0\right) x + \left(B_1 - B_0\right), \quad (14)$$

$$\rho_2 = \left(A_1 A_0 - A_0 A_1\right) x + \left(A_1 B_0 - A_0 B_1\right), \quad (15)$$

are linearly independent along the optimal trajectory. This would ensure that ϕ and $\dot{\phi}$ cannot simultaneously equal zero.[1] The number of times u switches value between 0 and 1 is equal to the number of times ϕ crosses zero.

Next, we investigate time-optimal trajectories in the three common dc-dc converter topologies. Following the approach above, we first study the linear independence of the vectors ρ_1 and ρ_2, and then the number of zero crossings of the switching function to determine the number of switches in a time-optimal trajectory.

IV. Analysis of the Dc-Dc Buck Converter

We first investigate the possibility of singular trajectories in the buck converter. Next, we determine the number of switches in a time-optimal trajectory.

A. Singular Trajectories

Substituting A_1, B_1 and A_0, B_0 from Table I in (14)-(15) we get

$$\rho_1 = \left[\frac{V_{IN}}{L}, 0\right]^{\mathrm{T}}, \quad \rho_2 = \left[0, -\frac{V_{IN}}{LC}\right]^{\mathrm{T}}. \quad (16)$$

By inspection, it is clear that ρ_1 and ρ_2 are linearly independent. Hence, singular trajectories are ruled out in the buck converter.

B. Number of Switches in a Time-Optimal Trajectory

Substituting A_1, B_1 and A_0, B_0 from Table I in (10) yields the following expression for the switching function

$$\phi_{\mathsf{buck}}(t) = \frac{V_{IN}}{L} p_1(t). \quad (17)$$

To study the switching function further, we need to investigate the evolution of $p_1(t)$. From (12), we see that the costates evolve according to

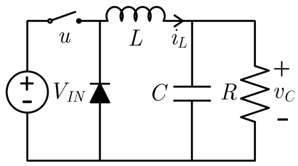
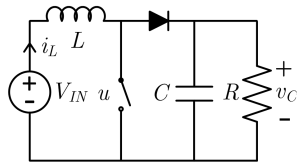
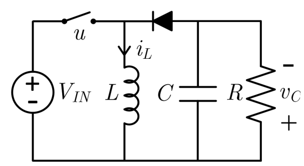
$$\begin{aligned} \frac{dp_1}{dt} &= -\frac{1}{C} p_2, \\ \frac{dp_2}{dt} &= \frac{1}{L} p_1 + \frac{1}{RC} p_2. \end{aligned} \quad (18)$$

The above equations can be combined to yield the following second-order differential equation that governs p_1:

$$\frac{d^2 p_1}{dt^2} - \frac{1}{RC}\frac{dp_1}{dt} + \frac{1}{LC} p_1 = 0. \quad (19)$$

[1]This argument holds for the two-dimensional state-space models adopted for the dc-dc converter dynamics in this work. For higher-order systems, or if ρ_1 and ρ_2 are not linearly independent, higher-order derivatives of ϕ need to be investigated [15].

Table I
DC-DC CONVERTER STATE-SPACE MODELS

Buck Converter	Boost Converter	Buck-Boost Converter
		
$A_1 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}, B_1 = \begin{bmatrix} \frac{V_{IN}}{L} \\ 0 \end{bmatrix}$	$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix}, B_1 = \begin{bmatrix} \frac{V_{IN}}{L} \\ 0 \end{bmatrix}$	$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix}, B_1 = \begin{bmatrix} \frac{V_{IN}}{L} \\ 0 \end{bmatrix}$
$A_0 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}, B_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$A_0 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}, B_0 = \begin{bmatrix} \frac{V_{IN}}{L} \\ 0 \end{bmatrix}$	$A_0 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}, B_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

The eigenvalues corresponding to the solution of (19) are

$$\lambda_{1,2} = \frac{1}{2RC} \pm \frac{1}{2}\sqrt{\left(\frac{1}{RC}\right)^2 - 4\left(\frac{1}{LC}\right)}. \tag{20}$$

The nature of the eigenvalues determines the evolution of p_1 and hence of ϕ. Three cases need to be investigated.

Real and distinct eigenvalues: This corresponds to the case where the eigenvalues in (20) are such that: $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1, \lambda_2 > 0$. The solution to (19) is of the form

$$p_1(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \tag{21}$$

where c_1 and c_2 are constants that can be determined from boundary conditions. From (17), we see that the switching function is given by

$$\phi_{\text{buck}}(t) = \frac{V_{IN}}{L}\left(c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\right). \tag{22}$$

To investigate the number of zero crossings of $\phi_{\text{buck}}(t)$, factor out $e^{\lambda_1 t}$ as follows:

$$\phi_{\text{buck}}(t) e^{-\lambda_1 t} = \frac{V_{IN}}{L}\left(c_1 + c_2 e^{(\lambda_2 - \lambda_1)t}\right). \tag{23}$$

Differentiating above with respect to time,

$$\frac{d}{dt}(\phi_{\text{buck}}(t) e^{-\lambda_1 t}) = \frac{1}{L} V_{IN} c_2 (\lambda_2 - \lambda_1) e^{(\lambda_2 - \lambda_1)t} \neq 0. \tag{24}$$

Therefore $\phi_{\text{buck}}(t) e^{-\lambda_1 t}$ has at most one root. Since $e^{-\lambda_1 t} > 0$, we can conclude that $\phi_{\text{buck}}(t)$ has at most one zero crossing.

Real repeated eigenvalues: This corresponds to the case where the eigenvalues in (20) are such that: $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$, $\lambda > 0$, which represents the load $R = \frac{1}{2}\sqrt{\frac{L}{C}}$. In this case, the general solution to (19) is of the form

$$p_1(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}, \tag{25}$$

where c_1 and c_2 are constants that can be determined from boundary conditions. From (17), we see that the switching function is given by

$$\phi_{\text{buck}}(t) = \frac{V_{IN}}{L}\left(c_1 e^{\lambda t} + t c_2 e^{\lambda t}\right). \tag{26}$$

As before, to investigate the number of zero crossings of $\phi_{\text{buck}}(t)$, factor out $e^{\lambda t}$ as follows:

$$\phi_{\text{buck}}(t) e^{-\lambda t} = \frac{V_{IN}}{L}\left(c_1 + t c_2\right). \tag{27}$$

Differentiating with respect to time,

$$\frac{d}{dt}(\phi_{\text{buck}}(t) e^{-\lambda t}) = \frac{1}{L} V_{IN} c_2 \neq 0. \tag{28}$$

As before, this implies $\phi_{\text{buck}}(t) e^{-\lambda t}$ has at most one root. Since $e^{-\lambda t} > 0$, we can conclude that $\phi_{\text{buck}}(t)$ has at most one zero crossing.

Imaginary eigenvalues: This corresponds to the case where the eigenvalues in (20) are such that: $\lambda_1 = \alpha + j\beta$, $\lambda_2 = \alpha - j\beta$, $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta > 0$. The general solution to (19) then yields the following switching function:

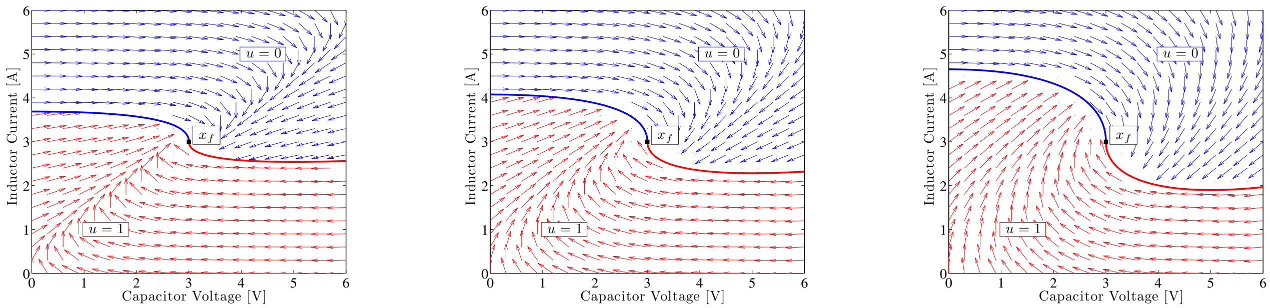
$$\phi_{\text{buck}}(t) = \frac{V_{IN}}{L} c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t), \tag{29}$$

from which it is clear that no claims can be made regarding the number of zero crossings of $\phi_{\text{buck}}(t)$. Therefore, when the dynamics are governed by complex eigenvalues, the maximum principle does not yield any further insight into the number of switches involved in a time-optimal trajectory.

In conclusion, for the buck converter, Pontryagin's maximum principle demonstrates that a time optimal transfer of converter states involves at most one change in the switch state when the system eigenvalues are real. The maximum principle does not yield any insight into the number of switches when the eigenvalues are imaginary.

C. Numerical Simulation

System trajectories for a representative buck converter with real, real identical, and imaginary eigenvalues are plotted in Figs. 1(a), 1(b), and 1(c), respectively. Let x_f be the desired final state, then the *switching curve* is the set of points such that the trajectory intersects x_f while maintaining one switch state, either $u = 0$ or $u = 1$. When the initial state is above the switching curve, setting $u = 0$ until the trajectory hits the switching curve and then changing to $u = 1$ (vice versa for states below the switching curve) achieves time-optimal performance for the real eigenvalue cases, as in Figs. 1(a) and



(a) $L = 2$ H, $C = \frac{1}{4}$ F, $R = 1\ \Omega$ (b) $L = \frac{1}{2}$ H, $C = \frac{1}{8}$ F, $R = 1\ \Omega$ (c) $L = \frac{1}{2}$ H, $C = \frac{1}{4}$ F, $R = 1\ \Omega$

Figure 1. State plane and switching curve for a buck converter with (a) distinct real, (b) identical real, and (c) imaginary eigenvalues. The bold line is the switching curve, in which the system will reach the desired operating point, x_f. To reach the desired final operating point from a point above the switching curve, the converter active switch is first set to $u = 0$. Similarly, to reach the desired final operating point from a point below the switching curve, the converter active switch is first set to $u = 1$.

1(b). Between the three different cases, the trajectories and switching surface vary slightly, but the trends generally remain the same. This might suggest that a single switch action is optimal in the buck converter where the system eigenvalues are imaginary. However, as shown above, the maximum principle does not yield any further insights in this case.

V. Analysis of the Dc-Dc Boost and Buck-Boost Converters

In this section, we apply the maximum principle to study time-optimal trajectories in boost and buck-boost converters. Consider first, the boost converter. As before, we first evaluate the possibility of singular trajectories by investigating the linear independence of the vectors ρ_1 and ρ_2. Substituting A_1, B_1 and A_0, B_0 from Table I in (14)-(15) we get

$$\rho_1 = \left[\frac{v_C}{L}, -\frac{i_L}{C}\right]^{\mathrm{T}}, \tag{30}$$

$$\rho_2 = \left[-\frac{v_C}{LRC}, -\frac{i_L}{RC^2} - \frac{V_{IN}}{LC}\right]^{\mathrm{T}}. \tag{31}$$

It is straightforward to show that ρ_1 and ρ_2 are linearly dependent if $V_{IN} = -\frac{2Li_L}{RC}$. However, this is impossible if the inductor current $i_L > 0$ along the optimal trajectory. Hence, singular trajectories are not obtained in an ideal boost converter with $i_L > 0$, $v_C > 0$.

Similarly, for the buck-boost converter, substituting A_1, B_1 and A_0, B_0 from Table I in (14)-(15) we get

$$\rho_1 = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix} x + \begin{bmatrix} \frac{V_{IN}}{L} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{v_C}{L} + \frac{V_{IN}}{L} \\ -\frac{i_L}{C} \end{bmatrix}, \tag{32}$$

$$\begin{aligned} \rho_2 &= \begin{bmatrix} 0 & -\frac{1}{LRC} \\ -\frac{1}{RC^2} & 0 \end{bmatrix} x - \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} \frac{V_{IN}}{L} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{v_C}{LRC} \\ -\frac{i_L}{RC^2} - \frac{V_{IN}}{LC} \end{bmatrix}. \end{aligned} \tag{33}$$

Suppose ρ_1 and ρ_2 are linearly dependent so that $\rho_1 = \alpha\rho_2$. This would require

$$\begin{aligned} &\frac{v_C}{L} + \frac{V_{IN}}{L} = -\alpha\frac{v_C}{LRC} \\ &\implies \alpha = -RC\left(1 + \frac{V_{IN}}{v_C}\right), \end{aligned} \tag{34}$$

which further implies

$$\begin{aligned} &-\frac{i_L}{C} = \alpha\left(-\frac{i_L}{RC^2} - \frac{V_{IN}}{LC}\right) \\ &\implies \left(1 + \frac{V_{IN}}{v_C}\right)\left(1 + \frac{V_{IN}}{i_L}\frac{RC}{L}\right) = -1. \end{aligned} \tag{35}$$

But assuming that along the optimal trajectory, $i_L > 0$ and $v_C > 0$, then the condition above cannot be satisfied. This proves that ρ_1 and ρ_2 are linearly independent and singular trajectories are not obtained.

Now consider applying the same strategy as the buck converter to study the number of switches in these topologies. Towards this end, substituting A_1, B_1 and A_0, B_0 from Table I in (10) yields the following expression for the switching function in the boost converter:

$$\phi_{\text{boost}}(t) = \frac{p_1(t)v_C(t)}{L} - \frac{p_2(t)i_L(t)}{C}. \tag{36}$$

Similarly, for the buck-boost converter, we get:

$$\phi_{\text{buck-boost}}(t) = \frac{p_1(t)\left(v_C(t) + V_{IN}\right)}{L} - \frac{p_2(t)i_L(t)}{C}. \tag{37}$$

The challenge in analyzing the functions above is that the evolution of $\phi_{\text{boost}}(t)$ and $\phi_{\text{buck-boost}}(t)$ depends on the value of u (notice, that in the buck converter, $\phi_{\text{buck}}(t)$ just depended on $p_1(t)$ that was independent of the value of $u(t)$). Therefore, it is unclear how to proceed in applying the maximum principle. One option (that has unfortunately yielded limited success) is to separately consider the two cases $u = 0$ and $u = 1$. For each choice of u, we will denote the corresponding switching function by ϕ_0 and ϕ_1, respectively. If we could prove that ϕ_0 and ϕ_1 have at most one zero crossing, then this would imply that time-optimal trajectories in the boost and buck-boost converters involve at most one switch. While this is easy to show for the $u = 1$ case in both converters, results for $u = 0$ are very conservative, and not helpful. An alternate option that might yield further insight in these topologies relates to optimal-control approaches formulated for hybrid systems [15].

VI. Concluding Remarks and Open Questions

This work examined time-optimal trajectories in buck, boost, and buck-boost converters with Pontryagin's maximum principle. We ruled out the possibility of singular trajectories, and proved that single-switch control is time-optimal for the buck converter when the eigenvalues corresponding to the converter dynamics are real (distinct or identical). However, it was noted that the maximum principle provides limited insight into time-optimal trajectories in the boost, and buck-boost converters.

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