Optimal Tap Settings for Voltage Regulation Transformers in Distribution Networks

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***Abstract*—In this paper, we propose a method to optimally set the taps of voltage regulation transformers in distribution networks. Specifically, we cast the problem of optimally choosing the tap settings as a rank-constrained semidefinite program (SDP) with the transformer tap positions removed from the network's admittance matrix and replaced by additional constraints and optimization variables. Then, the non-convex rank-1 constraint that arises from this rank-constrained SDP formulation is relaxed, which leads to a convex SDP program. The tap positions are obtained from the primary- and secondary-side bus voltages yielded by the optimal solution. We present several case studies with a 14-bus single-phase and a 15-bus three-phase distribution system to demonstrate the validity of our method.**

***Index Terms*—Tap-Changing under-load (TCUL) transformer, Optimal Power Flow (OPF), Semidefinite Programming (SDP).**

I. Introduction

In power distribution systems, it is common to utilize tap-changing under-load (TCUL) transformers and switched capacitors to regulate voltage. In general, the settings of these devices are automatically adjusted via automatic voltage regulators (AVRs) that act based upon local voltage measurements. While this decentralized approach to voltage regulation is effective for voltage control purposes, it may not be optimal in the context of the overall system operation, e.g., it might not result in minimum network losses. In this regard, by relying on point-to-point communications between the various regulation devices and a centralized processor, it is possible to optimally decide the device settings; this decision-making problem is very much like the optimal power flow problem (OPF) that commonly arises in transmission networks.

In this paper, we tailor the OPF formulation to distribution networks to include the tap settings of distribution transformers as decision variables; this is crucial to enable optimal voltage regulation in distribution networks. The inclusion of transformer tap settings in the OPF formulation for transmission networks has been investigated for decades. For example, in [1], the transformer tap positions are included as discrete variables in the OPF problem, which becomes a mixed-integer program (MIP). Unfortunately, the computational complexity of this formulation grows exponentially as the number of transformers increases, and thus becomes intractable for large systems. To tackle this problem, it has been proposed to represent the transformer tap positions with continuous variables, and then round the solutions to the closest discrete valuables. This alternative approach can yield acceptable results without incurring the exponential complexity; see e.g., [1]–[3]. However, all these approaches are restricted to the standard OPF formulations, and thus may suffer from the convergence issues of traditional solvers.

Cast as a rank-constrained semidefinite program (SDP), the OPF problem can be solved using convex solvers by dropping the only non-convex rank-1 constraint; see e.g., [4]–[7]. In general, this rank relaxation is not guaranteed to attain the global minimum, in particular for mesh networks. Interestingly, it has been shown that under some mild conditions, the optimal solution for the relaxed SDP-based OPF problem turns out to be of rank 1 for tree networks typical of distribution systems [4]–[6]. This implies that the rank relaxation scheme is actually guaranteed to attain the global optimum of the original OPF problem. In addition to handling the OPF problem, the SDP-based approach also constitutes a very promising tool to tackle the nonconvexities in other operations and control tasks for distribution systems.

Leveraging the convex SDP formulation, the voltage regulation transformer tap positions can be incorporated in the OPF problem as decision variables by introducing a virtual secondary-side bus per transformer with additional constraints [8]. However, the voltage regulation transformer model proposed in [8] is insufficient. The reason is two-fold: (i) the optimal solution to the relaxed problem fails to yield a rank-1 matrix; and (ii) it is only valid for a single-phase network. The first issue arises from the fact that the network is broken into two disconnected subnetworks after introducing the virtual bus. The three-phase extension is very important for distribution systems, which fails due to the independent setting of the tap position for each phase in the SDP-based formulation. As it will become more clear later on, it is impossible to enforce the per-phase coupling between the primary- and secondary-side buses in the SDP-based formulation. To tackle the two issues discussed above, we propose to modify the model in [8] by including a highly resistive line between the primary- and secondary-side buses. This modification does not introduce additional complexity in the OPF problem and successfully resolves the two aforementioned issues.

The remainder of this paper is organized as follows. Section II introduces the system model, and formulates the transformer tap-setting problem. Section III discusses the rank issue of the solution from Section II. The modified transformer model is introduced in Section IV. Section V presents the case studies and the paper is concluded with Section VI.

II. PRELIMINARIES AND PROBLEM FORMULATION

In this section, we first provide an overview of the positive semidefinite formulation of the power flow equations and introduce the transformer model proposed in [8]. Then, we formulate the optimal voltage regulation transformer tap-setting problem as a relaxed SDP optimization problem.

A. Power System Model

Consider an n-bus power system that has n_t transformers with the ability to adjust their tap positions. Let $\mathcal{N}$ denote the set of buses $\{1,\ldots,n\}$ and $\mathcal{T}$ be the set of transformers $\{1,\ldots,n_t\}$. The set of buses incident to the primary-side of a transmission line containing a transformer is $\mathcal{N}_p := \{p_t \in \mathcal{N} \,|\, t \in \mathcal{T}\}$; and similarly for the set of buses incident to the secondary side we have that $\mathcal{N}_s := \{s_t \in \mathcal{N} \,|\, t \in \mathcal{T}\}$. The edge-set that represents the set of transmission lines (single-, two-, or three-phase) is $\mathcal{E} := \{(i,k) \,|\, i,k \in \mathcal{N}\} \subseteq \mathcal{N} \times \mathcal{N}$; and the set of edges with transformers is denoted by $\mathcal{E}_t := \{(p_t, s_t) \in \mathcal{E} \,|\, t \in \mathcal{T}\} \subseteq \mathcal{E}$.

Let V_i denote the i^{th} entry of the bus voltage vector[1] $v \in \mathbb{C}^n$, and $\mathcal{H}_i := \{i\} \cup \{k \,|\, (i,k) \in \mathcal{E}\}$. The complex power injection at bus $i \in \mathcal{N}$ is given by

$$S_i = V_i \sum_{k \in \mathcal{H}_i} Y_{ik}^* V_k^*, \tag{1}$$

where Y_{ik} is the (i,k) entry of the admittance matrix Y. We define $W \in \mathbb{C}^{n \times n}$ as the outer product of v:

$$W = vv^H = \begin{bmatrix} |V_1|^2 & V_1 V_2^* & \cdots & V_1 V_n^* \\ V_1^* V_2 & |V_2|^2 & \cdots & V_2 V_n^* \\ \vdots & \vdots & \ddots & \vdots \\ V_1^* V_n & V_2^* V_n & \cdots & |V_n|^2 \end{bmatrix}; \tag{2}$$

this implies that W must be positive semidefinite (PSD) ($W \succeq \mathbf{0}$) with rank 1. This way, the complex power in (1) is linearly related to the entries of W as follows

$$S_i = \mathrm{Tr}\left(H_i W\right), \tag{3}$$

with $H_i := Y^H E_i$, where $E_i := e_i e_i^{\mathrm{T}}$ is the outer product of the Kronecker vector e_i with all entries zero except the i^{th} one. Further defining $E_{ik} := (e_i - e_k)\, e_i^{\mathrm{T}}$, the complex power flowing from bus i to k over line $(i,k) \in \mathcal{E}$ is given by

$$S_{ik} = \mathrm{Tr}\left(A_{ik} W\right), \tag{4}$$

where $A_{ik} := -e_k^{\mathrm{T}} Y^H e_i E_{ik}$.

B. Ideal Voltage Regulation Transformer Model

Consider the simple 3-bus network in Fig. 1, comprised of two transmission lines, constant PQ loads, and a voltage regulation transformer attached to bus p_t. It is assumed that the primary-side of the regulator is always closest to the distribution feeder.

The transformer tap position $\eta_t \in \{\underline{\eta}, \overline{\eta}\}$ is a discrete variable with 32 values, where each tap change corresponds to a 5/8% change of nominal voltage and $\{\underline{\eta}, \overline{\eta}\}$ are the minimum and maximum percentages of nominal voltage, respectively [9]. Transformers are commonly replaced by a two-port block whose impedance is dependent on the tap position and appears nonlinearly in the admittance matrix of the equivalent circuit model [8]. Figure 2 shows the modified circuit used to remove the nonlinearity of the transformer tap position. We introduce a virtual bus s_t' on the secondary-side of the ideal transformer. Typically, core losses on distribution system voltage regulation transformers are generally ignored [10]. However, one can refer the voltage regulation transformer impedance to the secondary side and include it with the line parameter $y_{p_t s_t}$. The set of buses now becomes

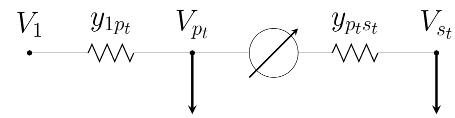

Figure 1: 3-Bus System

$$\overline{\mathcal{N}} = \mathcal{N} \cup \mathcal{N}_{s'}, \tag{5}$$

where $\mathcal{N}_{s'} := \{s_t' \,|\, s_t \in \mathcal{N}_s\}$ is the set of virtual buses added to the secondary side of each transformer. The set of edges is further modified as

$$\overline{\mathcal{E}} = \{\mathcal{E} \backslash \mathcal{E}_t\} \cup \{(s_t', s_t) \,|\, t \in \mathcal{T}\}. \tag{6}$$

We also update the edge-set for the transformers as $\overline{\mathcal{E}}_t := \{(p_t, s_t') \,|\, t \in \mathcal{T}\}$. Note that $\overline{Y}_{s_t' s_t} = Y_{p_t s_t}$ and $\overline{Y}_{p_t s_t} = 0$ in the updated admittance matrix.

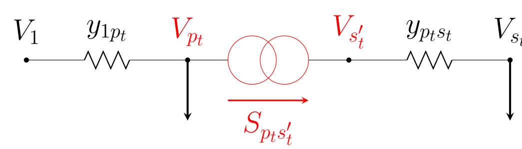

Figure 2: Equivalent 3-Bus System

In the transformer model, we assume that the network is electrically disconnected at (p_t, s_t'); thus, we capture the power $S_{p_t s_t'}$ through the transformer by introducing a PQ load and source at buses p_t and s_t', respectively. The tap ratio can be eliminated by adding the following constraints

$$\underline{\eta}^2 \left|V_{p_t}\right|^2 \le \left|V_{s_t'}\right|^2 \le \overline{\eta}^2 \left|V_{p_t}\right|^2 \tag{7a}$$

$$\mathrm{Re}\left\{V_{p_t}\right\} \times \mathrm{Im}\left\{V_{s_t'}\right\} \;=\; \mathrm{Re}\left\{V_{s_t'}\right\} \times \mathrm{Im}\left\{V_{p_t}\right\} \tag{7b}$$

$$\mathrm{Re}\left\{V_{p_t}\right\} \times \mathrm{Re}\left\{V_{s_t'}\right\} \;\ge\; 0 \tag{7c}$$

$$\mathrm{Im}\left\{V_{p_t}\right\} \times \mathrm{Im}\left\{V_{s_t'}\right\} \;\ge\; 0, \tag{7d}$$

where the last three constraints (7b)–(7d) ensure that the transformer is not a phase-shifter, which is equivalent to $W_{p_t s_t'} = W_{s_t' p_t} \ge 0$ for the matrix W. Once the solution is determined, the tap position of transformer $t \in \mathcal{T}$ can be obtained by $\eta_t = \sqrt{V_{s_t'}^2 / V_{p_t}^2} = \sqrt{W_{s_t' s_t'} / W_{p_t p_t}}$.

When considering the updated transformer model, the power flow equations in (3) are modified as follows:

[1] The notation in this section is for single-phase, V_{ia}, V_{ib}, V_{ic} will be used to distinguish the phases on bus i for a three-phase network.

1) No Transformer: If there is no transformer incident to bus i, then the power flow equation remains unchanged and is

$$S_i = S_i^{gen} - S_i^{ld} = \mathrm{Tr}\left(H_i W\right), \quad \forall i \in \overline{\mathcal{N}} \setminus \left\{\mathcal{N}_p \cup \mathcal{N}_{s'}\right\}, \tag{8}$$

where the generation S_i^{gen} and load S_i^{ld} are positive quantities.

2) Ideal voltage regulation transformer model: Since the tap position η_t for $t \in \mathcal{T}$ is not included in the formulation of the admittance matrix, we compare the input/out power of the ideal transformer. If bus i is incident to the primary-side of a transformer, then the power flow equation becomes

$$S_{p_t} - S_{p_t s_t'} = \mathrm{Tr}\left(H_{p_t} W\right), \quad \forall p_t \in \mathcal{N}_p, \tag{9}$$

or if it is incident to the secondary side, then we have

$$S_{p_t s_t'} = \mathrm{Tr}\left(H_{s_t'} W\right), \quad \forall s_t' \in \mathcal{N}_{s'}. \tag{10}$$

Notice that all of the H matrices here correspond to the updated admittance matrix $\overline{Y}$.

C. Problem Formulation

Inspired by the rank relaxation technique for OPF as formally discussed in [5]–[7], one can drop the rank-1 condition on W that arises from (2) and formulate the convex transformer tap position problem as

$$\min_{W \succeq \mathbf{0}, S_{p_t s_t'}} \mathrm{Re}\left\{\mathrm{Tr}\left(A_{12} W\right)\right\} \tag{11}$$

such that

$$\mathrm{Tr}\left(H_i W\right) - S_i = 0, \quad \forall i \in \overline{\mathcal{N}} \setminus \left\{\mathcal{N}_p \cup \mathcal{N}_{s'}\right\} \tag{12}$$

$$\mathrm{Tr}\left(H_{p_t} W\right) - S_{p_t} + S_{p_t s_t'} = 0, \forall p_t \in \mathcal{N}_p, (p_t, s_t') \in \overline{\mathcal{E}}_t \tag{13}$$

$$\mathrm{Tr}\left(H_{s_t'} W\right) - S_{p_t s_t'} = 0, \forall s_t' \in \mathcal{N}_{s'}, (p_t, s_t') \in \overline{\mathcal{E}}_t \tag{14}$$

and

$$\underline{V}^2 \leq \ W_{ii} \ \leq \overline{V}^2, \qquad \forall i \in \overline{\mathcal{N}} \setminus \mathcal{N}_{s'} \tag{15}$$

$$\underline{\eta}^2 W_{p_t p_t} \leq W_{s_t' s_t'} \leq \ \overline{\eta}^2 W_{p_t p_t}, \quad \forall \left(p_t, s_t'\right) \in \overline{\mathcal{E}}_t \tag{16}$$

and

$$W_{p_t s_t'} = W_{s_t' p_t} \geq 0, \quad \forall \left(p_t, s_t'\right) \in \overline{\mathcal{E}}_t. \tag{17}$$

The cost function in (11) minimizes the generation required from the transmission network through the feeder. This choice for the objective functions follows naturally since power generation from the utilities is generally not located at the distribution level. Line flow constraints are excluded for simplicity, but can easily be included as additional inequality constraints, see e.g., [5]–[7].

III. Single-Phase Rank Condition

This section aims to show why the ideal transformer model in Section II-C fails to yield a rank-1 solution to the problem in (11)–(17) even for single-phase tree networks.

Suppose that the system contains a single voltage regulation transformer, which introduces a non-overlapping two-area partition for the augmented network $(\overline{\mathcal{N}}, \overline{\mathcal{E}})$. Let $W^{(1)}$ and $W^{(2)}$ denote the two submatrices of W corresponding to the two-area partition induced. Hence, the optimization cost function in (11) turns out to be only dependent on $W^{(1)}$. Moreover, all the constraints in (12)–(14) only couple the power flow quantities within each area and also the transformer power flow variable $S_{p_t s_t'}$. The entry-wise voltage bound constraints in (15) are local for each area, while only the transformer voltage regularization constraint (16) would involve entries from both submatrices. This way, it is possible compactly write the local constraints in (12)–(15) as follows:

$$W^{(1)} \in \mathcal{C}^{(1)}\{S_{p_t s_t'}\} \tag{18}$$

$$W^{(2)} \in \mathcal{C}^{(2)}\{S_{p_t s_t'}\}, \tag{19}$$

where both set constraints are defined by the power injection quantities within each area and $S_{p_t s_t'}$.

Proposition 1: The convex SDP relaxation problem with matrix W can be recast as one involving its three submatrices:

$$\min_{W^{(1)}, W^{(2)}, W_{p_t s_t'}, S_{p_t s_t'}} \mathrm{Re}\left\{\mathrm{Tr}\left(A_{12}^{(1)} W^{(1)}\right)\right\}, \tag{20}$$

subject to (16)–(19), and the following PSD constraints

$$W^{(1)} \succeq \mathbf{0}, W^{(2)} \succeq \mathbf{0}, \begin{bmatrix} W_{p_t p_t} & W_{p_t s_t'} \\ W_{p_t s_t'} & W_{s_t' s_t'} \end{bmatrix} \succeq \mathbf{0}. \tag{21}$$

Proof: In order to prove the result, we leverage the results in [11] for completing a partial Hermitian matrix to a full PSD one by using the so-termed graph "chordal" property. A chordal graph has no minimal cycles with number of nodes greater than 3. To this end, construct a graph $\tilde{\mathcal{G}}$ with the node set $\overline{\mathcal{E}}$, with all its edges corresponding to the off-diagonal entries in the three submatrices in (21). It is possible to show that this graph $\tilde{\mathcal{G}}$ is chordal; and the reasons are two-fold. First, there is no cycle that contain nodes from both two areas as defined by the voltage regulation transformer partition. Second, all the minimum cycles within each area have only three nodes since every subnetwork forms a complete graph. Interestingly, for the chordal graph $\tilde{\mathcal{G}}$, the submatrices in (21) correspond to its three maximal cliques. The Hermitian matrix completion results in [11] establish that if all the given entries in matrix W induce a chordal graph with all submatrices corresponding to the graph's maximum cliques are PSD, then it is possible to complete W to a PSD matrix. Leveraging this claim, one can form a full PSD matrix W with its three PSD submatrices in (21) given. Since the completed W would be a feasible solution to the original problem (11), the reformulated problem (20) attains the minimum objective no smaller than the original one (11).

Reversely, the PSD property of W implies that all its submatrices are PSD, including those three in (21). Hence, the original problem (11) would at least attain a minimum cost no smaller than that of (20).

Based on the two arguments above, the equivalence of the two problems follows easily since they attain the same objective at the optimum. ■

The constraint $W_{p_t s_t'} \geq 0$ for the reformulated problem in Proposition 1, together with the PSD condition for the 2×2 submatrix in (21), yields the following equivalent condition

$$0 \leq W_{p_t s_t'} \leq \sqrt{W_{p_t p_t} W_{s_t' s_t'}}, \tag{22}$$

where the rank-1 of W is satisfied only if the right-side strict equality holds. Since the variable $W_{p_t s'_t}$ is solely present in the condition in (22), any optimal solution to the reformulated problem can be easily completed to a matrix W with rank greater than 1, by setting $W_{p_t s'_t} = 0$ without sacrificing the optimality. This way, the corresponding solution to the original problem (11) would be of at least rank 2. Next, we provide an example to demonstrate the issues with non-rank-1 solutions to the problem in (11)–(17).

Example 1: Consider the 3-bus network in Fig. 1, and further assume that it is a lossless system and the voltage regulation transformer is ideal. The primary side of the transformer is the bus 2 with no load, while the secondary-side bus 3 has the only load with the known complex value S_3. The lossless network assumption implies that $y_{12} = y_{23} = 0$. With the ideal voltage regulation transformer and $y_{23} = 0$, it is possible to simply merge the virtual bus $3'$ with the original load bus 3 in the equivalent circuit in Fig. 2. It is easy to show that the minimum cost in (11) equals to $\mathrm{Re}\{S_2\}$, with no system loss and the transformer power flow $S_{23} = S_3$. Interestingly, the tap position is not essential for attaining the minimum cost, hence for simplicity consider the case where $W_{22} = W_{33} = 1$. Similarly, the variable W_{23} does not affect the achievable objective either, so it can take any value in the interval $[0, 1]$ to satisfy the PSD constraint and (17), as given in (22). There exist multiple optimal solutions with $0 \leq W_{23} < 1$ but those corresponding W matrices would have rank of at least 2. This simple example demonstrates the rank issue in the problem in (11)–(17). □

Although this section only discusses the case of a single transformer, it is possible to extend to multiple transformers where the rank of the optimal W would increase as well. This is possible by partitioning the system into (n_t+1) areas for n_t transformers, and showing that the constraint (17) would fail to satisfy the rank condition for any two neighboring areas. In the next section, we intend to solve this problem by introducing a modified (lossy) voltage regulation transformer model.

IV. Non-Ideal Voltage Regulation Transformer Model

Before introducing our model, we will first discuss some issues pertaining the ideas discussed in Section III to three-phase networks. Suppose that we have a single-phase network with $n_t \geq 2$. We can solve for $W^{(1)}, \ldots, W^{(n_t)}$, where each area will have its own local slack bus and the correct phase angles can be computed recursively downstream of the feeder using the boundary condition $\mathrm{Angle}\{V_{p_t}\} = \mathrm{Angle}\{V_{s'_t}\}\ \forall t \in \mathcal{T}$. Now, consider a three-phase network; then, the constraint in (17) prevents a phase shift across the transformer on each phase. It is infeasible to enforce the correct phase angle offset for the three phases of the slack bus at each area. This way, the incorrect three-phase angle offsets would propagate in the rest of the area, which prevents us from obtaining a meaningful solution first and then correcting it with the aforementioned the recursive scheme. Hence, unlike the single-phase system, the angles between the phases have to be enforced on each side of the transformer.

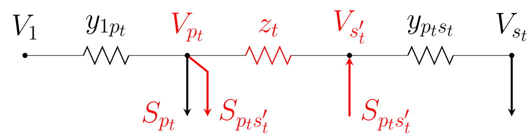

Figure 3: Equivalent 3-Bus System

To address this problem, we propose to modify the original model that removes the transformer tap positions from the power flow constraints in the minimization problem. As illustrated in Fig. 3, rather than replacing the voltage regulation transformer with an ideal transformer, we instead introduce a virtual transmission line that has a large impedance z_t, the choice of which is discussed later. The power transferred through transformer $S_{p_t s'_t}$ remains as the load or source to be optimized on the primary or secondary side of the transformer, respectively. As the impedance $|z_t| \to \infty$, the proposed model here would reduce to the original one discussed in Section II. Before reaching this limiting case, the system remains electrically connected and there will be a current through (p_t, s'_t), which is the scenario of interest to us. As demonstrated by the numerical tests, the results of which are discussed in Section V, one can ensure that the actual power flow will almost mimic the original electrically disconnected model with the appropriate value of z_t; i.e., $S_{p_t s'_t}$ captures almost all of power flow through transformer t. More importantly, it is possible to maintain the phase angles on both sides of the transformer, as $\theta_{p_t} \approx \theta_{s'_t}$. This is highly attractive since the modified transformer model in essence allows us to solve an equivalent OPF problem as the original model, while enforcing the correct phase shift for the transformers. Notice that the set of buses would stay the same as in (5), and the transmission lines will include the newly introduced virtual lines too, as given by

$$\overline{\mathcal{E}} = \{\mathcal{E} \backslash \mathcal{E}_t\} \cup \{(s'_t, s_t)\,|\, t \in \mathcal{T}\} \cup \overline{\mathcal{E}}_t. \tag{23}$$

A. Transformer Impedance

Consider a simple 2-bus system with $V_1\angle\theta_1$, $V_2\angle\theta_2$, and $z = r + jx$, where bus 1 is the primary side of the transformer. The power loss on the line is given by

$$\begin{aligned} S_{loss} &= \frac{1}{z^*} V^2 \\ &= \frac{r + jx}{r^2 + x^2} \left(V_1^2 + V_2^2 - 2V_1 V_2 \cos\left(\theta_1 - \theta_2\right)\right). \end{aligned} \tag{24}$$

Since we are not working with a phase-shifting transformer, we can simplify the expression above by assuming that $\theta_1 \approx \theta_2$. Then, by defining the turns ratio as $a = N_2/N_1$, we obtain that

$$P_{loss} = \frac{r}{r^2 + x^2} V_1^2 \left(a^2 - 2a + 1\right), \tag{25}$$

$$Q_{loss} = \frac{x}{r^2 + x^2} V_1^2 \left(a^2 - 2a + 1\right), \tag{26}$$

where $a \in [0.9, 1.1]$ for the per unit voltages. In the case studies in Section V, we minimize generation of active power losses and choose $r \gg 0$ and $x = 0$. For a three-phase system, if z_t is too large, then the system behaves as an open circuit and yields the incorrect results discussed in Section II and III. Otherwise, if z_t is too small, then the power loss on the line will no longer be negligible. In numerical simulations, we observed that selecting the value of r to be a couple orders of magnitude larger than that of the neighboring lines would lead to the best results.

B. Problem Formulation

The constraints on the off-diagonal entries in (17) are no longer required with the system electrically connected; thus, the relaxed convex problem becomes

$$\min_{W \succeq \mathbf{0}, S_{p_t s'_t}} \operatorname{Re}\{\operatorname{Tr}(A_{12} W)\} \tag{27}$$

such that

$$\operatorname{Tr}(H_i W) - S_i = 0, \quad \forall i \in \overline{\mathcal{N}} \setminus \{\mathcal{N}_p \cup \mathcal{N}_{s'}\} \tag{28}$$

$$\operatorname{Tr}(H_{p_t} W) - S_{p_t} + S_{p_t s'_t} = 0, \forall p_t \in \mathcal{N}_p, (p_t, s'_t) \in \overline{\mathcal{E}}_t \tag{29}$$

$$\operatorname{Tr}\left(H_{s'_t} W\right) - S_{p_t s'_t} = 0, \forall s'_t \in \mathcal{N}_{s'}, (p_t, s'_t) \in \overline{\mathcal{E}}_t \tag{30}$$

and

$$\underline{V}^2 \leq \ W_{ii} \ \leq \overline{V}^2, \qquad \forall i \in \overline{\mathcal{N}} \setminus \mathcal{N}_{s'} \tag{31}$$

$$\underline{\eta}^2 W_{p_t p_t} \leq W_{s'_t s'_t} \leq \overline{\eta}^2 W_{p_t p_t}, \quad \forall (p_t, s'_t) \in \overline{\mathcal{E}}_t. \tag{32}$$

If a cost function different from the one in (27) is selected that does not maximize $|S_{p_t s'_t}|$, then we can introduce an additional constraint on the power transferred through the line (p_t, s'_t) as follows:

$$\left|\operatorname{Tr}(A_{p_t s'_t} W)\right| \leq \varepsilon, \tag{33}$$

where $\operatorname{Tr}(A_{p_t s'_t} W)$ should be real and $\varepsilon \approx 0$ is some tolerance for the optimization problem. Note that this cannot be a strict equality constraint because there will be a power loss through this line when $\eta_t \neq 1$.

V. Case Studies

The case studies presented in this section are derived from the IEEE 13-bus, three-phase, unbalanced distribution system given in [12]. Figure 4 shows the 15-bus three-phase topology used. Buses 650 and 651 were added between the Feeder and the voltage regulation transformer so that the system partition did not occur at the slack bus. Bus 693 was added to account for the distributed load along line (632, 671), and bus 692 was removed since it corresponded to a closed switch connected between buses 671 and 675.

A. Single-Phase Case

The single-phase 14-bus system is obtained from phase C only since it is present on every bus except for bus 652. For all three cases, the objective function used for the minimization problem is

$$\min_{W, S_{p_t s'_t}} \operatorname{Re}\{\operatorname{Tr}(A_{\text{Feeder},651} W)\}, \tag{34}$$

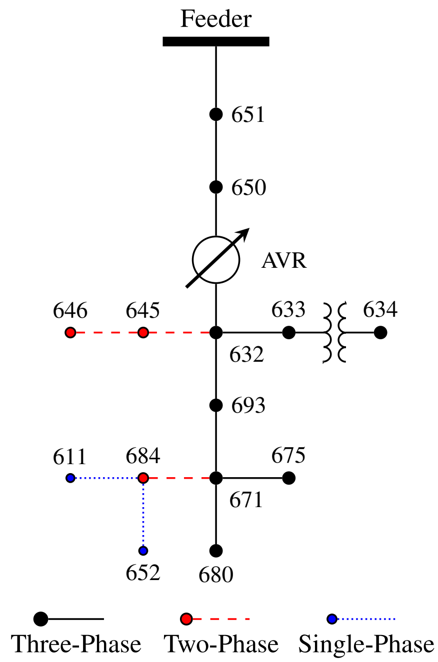

Figure 4: 15 Bus Distribution Network

which is the power transferred from the feeder across the transmission line (Feeder, 651).

The results are shown in Table I; as expected, the rank of W from the relaxed formulation in Section II does not satisfy the rank-1 constraint of the unrelaxed problem, i.e., $\operatorname{rank}(W) = 2$. However, when the system is partitioned around the transformer, we obtain the same W with the submatrices $W^{(1)}$ and $W^{(2)}$, as defined in Section III satisfying the rank-1 condition. The proposed formulation in Section IV produces a rank-1 solution with a 0.03% change in the objective function and a 0.4% change in the power sinked/sourced at the primary/secondary sides of the transformer. There were no changes in the bus voltages and tap positions obtained by any method. The impedance chosen for z_t was purely resistive and its magnitude was 100 times that of the neighboring transmission lines. The power $|S_t|$ transferred through z_t was 0.003 p.u., which $|S_t| \ll |S_{p_t s'_t}|$.

B. Three-Phase Case

The transformer model in Section IV is the only model applicable to the three-phase simulation. In this case, z_t is a purely real diagonal matrix whose entries have a magnitude 100 times larger than that of their neighboring transmission lines. The objective function

$$\min_{W \succeq \mathbf{0}, S_{p_t s'_t}} \sum_{i \in \mathcal{K}} \operatorname{Re}\{\operatorname{Tr}(H_i W)\} \tag{35}$$

was used to minimize the three-phase power at the feeder, where $\mathcal{K} = \{\text{Feeder}^{(a)}, \text{Feeder}^{(b)}, \text{Feeder}^{(c)}\}$. This expression was chosen to account for the power transferred through the mutual impedance across the the 4-wire transmission line.

Table I: SINGLE-PHASE CASE RESULTS

	Ideal Transformer	Partition Method	Non-Ideal Method
Rank of W	2	$W^{(1)} \Rightarrow 1$, $W^{(2)} \Rightarrow 1$	1
Objective [p.u.]	8.5776	8.5776	8.5806
V_{p_t} [p.u.]	$1.006\angle-0.285°$	$1.006\angle-0.285°$	$1.006\angle-0.285°$
$V_{s'_t}$ [p.u.]	$1.088\angle-0.285°$	$1.088\angle-0.285°$	$1.088\angle-0.285°$
V_{s_t} [p.u.]	$1.05\angle-2.817°$	$1.05\angle-2.817°$	$1.05\angle-2.817°$
$\|S_{p_t s'_t}\|$ [p.u.]	9.219	9.219	9.255
$\|S_t\|$ [p.u.]	0	0	0.003
Tap Position	-13	-13	-13

Table II: THREE-PHASE CASE RESULTS

	Phase A	Phase B	Phase C
V_{p_t} [p.u.]	$1.009\angle-0.06°$	$1.007\angle-120.16°$	$1.008\angle119.77°$
$V_{s'_t}$ [p.u.]	$1.068\angle-0.06°$	$1.078\angle-120.16°$	$1.074\angle119.77°$
V_{s_t} [p.u.]	$1.050\angle-1.41°$	$1.008\angle-122.04°$	$1.045\angle117.70°$
$\|S_{p_t s'_t}\|$ [p.u.]	6.60	9.58	9.16
$\|S_t\|$ [p.u.]	0.002816	0.002893	0.002248
Tap Position	-9.32	-11.24	-10.52

The results are shown in Table II. The solution of W was rank 1, the cost function has a value of 23.48 p.u., and the continuous tap positions for the three-phase transformer are $\{-9.32, -11.24, -10.52\}$. We have also performed an extensive search by enumerating all possible combinations of the discrete tap values for all three phases. The minimum cost was attained at 23.47 with the tap positions being $\{-9, -11, -10\}$, for the three phases, respectively. Clearly, the tap values in Table II exhibit competitive performance compared to the best solution, while the computational complexity of our method stays gracefully with the number of transformers growing as opposed to the enumeration. This suggests that the tap positions should be truncated to the nearest discrete value rather than being rounded.

VI. CONCLUDING REMARKS

In this paper, we developed a method to optimally set the taps of voltage regulation transformers in distribution networks. We demonstrated the applicability of this method via numerical examples involving single- and three-phase test systems. The main advantage of the our method, with respecting to existing approaches, is that our method will return a rank-1 solution if it exists and can scale up to a three-phase network.

We do not have a proof yet for whether or not the SDP relaxation will work on three-phase tree networks in general, however, we intend to address this problem in future work.

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