Distributed Voltage Regulation of Islanded Inertialess AC Microgrids with Lossy Networks

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***Abstract*— In this paper, we propose a distributed controller for voltage regulation in islanded ac microgrids with lossy electrical lines. The stability analysis presented takes into account the controller in charge of regulating the frequency in the microgrid, i.e., we do not assume decoupled operation of the proposed voltage controller and the frequency controller. We show that the frequency controller achieves active power load sharing by driving the power flows to a certain manifold, while the voltage controller moves along this manifold and adjusts the voltage magnitudes at the inverter buses to maintain a healthy voltage profile throughout the microgrid, i.e., voltage magnitudes at all buses are within some prescribed limits. Since achieving voltage regulation and reactive power sharing are known to be conflicting objectives, in this paper, we pursue the voltage regulation objective. However, reactive power sharing can be recovered to some extent by tuning a certain parameter in the voltage controller.**

I. Introduction

In this work, we are interested in the well-known problem of regulating the frequency and voltage magnitude of each bus in an islanded ac microgrid under network constraints and time-varying loads so as to: (i) achieve some desired active power sharing among the generating resources, (ii) ensure that the frequency at each bus is at the nominal value, (iii) the voltage magnitudes at all buses are within some limits prescribed by operational considerations, and (iv) to some extent, achieve reactive power sharing. In this paper, we focus primarily on achieving (iii) instead of (iv) since, as discussed in [1], voltage regulation and reactive power sharing are fundamentally conflicting objectives, and in many cases, achieving these two objectives simultaneously is not possible. However, instead of achieving exact reactive power sharing, the proposed voltage controller achieves reactive power sharing within a certain accuracy. The considered system incorporates inverter-interfaced energy resources and PQ-type loads. We consider networks where (i) the inductance-to-resistance ratio is perfectly homogeneous across all electrical lines, and (ii) the inductance-to-resistance ratio varies across the electrical lines within a certain interval, which typically holds in practice.

The aforementioned problem has been considered in recent works, and the proposed solutions are implemented through centralized or distributed control architectures (see, e.g., [1]–[6], and the references therein). Since the failure of the central processor in a centralized architecture would bring the whole system down and communication bandwidth and range constraints are more restrictive in a centralized communication network, we are more interested in a distributed implementation of the proposed controller. Next, we discuss some of the existing distributed controllers presented in the literature.

In [1], the authors propose distributed-averaging proportional-integral (DAPI) controllers for frequency and voltage regulation based on sparse communication among neighboring generators, and provide a small-signal stability analysis. In [2], the authors propose a consensus-based distributed voltage control for small phase angle differences assuming loads are modeled by constant impedances. In [3], the authors present a secondary voltage controller based on a PI control that uses the difference between voltage reference and voltage average of the distributed generators.

In this paper, we adopt the frequency control strategy proposed in our earlier work in [7] for lossless microgrids assuming fixed voltage magnitudes, and extend it to the more general case when voltage magnitudes are no longer fixed. We also propose a voltage controller that achieves voltage regulation. To ensure stable operation of the two controllers, which are coupled, two main strategies are employed: (i) the voltage controller is executed when the power flows are close to a certain manifold; and (ii) the model used for designing the voltage controller is linearized around this manifold, which allows us to decouple the reactive power injections from the voltage phase angles. The voltage controller has an additional tuning parameter for achieving, to some extent, reactive power sharing.

II. Preliminaries

In this section, we present the inverter-based microgrid model adopted in this work, and formulate the control objectives to be achieved.

A. *Microgrid Dynamic Model*

Consider a collection of inverters and loads interconnected by a lossy power network. The topology of this network can be described by an undirected graph, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} = \mathcal{V}^{(I)} \cup \mathcal{V}^{(L)}$, where $\mathcal{V}^{(I)} = \{1, \dots, m\}$ denotes the set of nodes with an inverter, and $\mathcal{V}^{(L)} = \{m+1, \dots, n\}$ denotes the set of nodes with a load; and where $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, with $\{i, j\} \in \mathcal{E}$ if nodes i and j are electrically connected. Let $\mathcal{N}_i$ and δ_i respectively denote the set of neighbors and the degree of node i.

Assuming balanced three-phase operation, let $\theta_i(t)$ and $V_i(t)$ denote the phase angle and magnitude of the instantaneous voltage of a single phase at node $i \in \mathcal{V}$ in a rotating

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coordinate frame with rotational frequency ω^*, and define $\theta(t) = [\theta_1(t), \ldots, \theta_n(t)]^{\mathrm{T}}$ and $V(t) = [V_1(t), \ldots, V_n(t)]^{\mathrm{T}}$. Then, the net active power, $P_i'(\theta(t), V(t))$, and reactive power, $Q_i'(\theta(t), V(t))$, injected into the network via node $i \in \mathcal{V}$ are given by

$$P_i'(\theta(t), V(t)) = \sum_{j \in \mathcal{V}} V_i(t)V_j(t)(G_{ij}\cos(\theta_i(t) - \theta_j(t)) + B_{ij}\sin(\theta_i(t) - \theta_j(t))), \quad (1)$$

$$Q_i'(\theta(t), V(t)) = \sum_{j \in \mathcal{V}} V_i(t)V_j(t)(G_{ij}\sin(\theta_i(t) - \theta_j(t)) - B_{ij}\cos(\theta_i(t) - \theta_j(t))), \quad (2)$$

with $G_{ii} = -\sum_{j\in\mathcal{V}} G_{ij}$, $B_{ii} = -\sum_{j\in\mathcal{V}} B_{ij}$, $G_{ij} = -g_{ij}$ and $B_{ij} = -b_{ij}$, where $g_{ij} > 0$ and $b_{ij} < 0$ are, respectively, the series conductance and series susceptance of the line connecting nodes i and j.

Let $\zeta := \frac{1}{|\mathcal{E}|} \sum\limits_{\{i,j\}\in\mathcal{E}} \frac{B_{ij}}{G_{ij}}$; then, we use the following linear transformation (used in [8] to recover a model with purely inductive lines):

$$\begin{bmatrix} P_i(\theta(t), V(t)) \\ Q_i(\theta(t), V(t)) \end{bmatrix} = \begin{bmatrix} \sin(\psi) & -\cos(\psi) \\ \cos(\psi) & \sin(\psi) \end{bmatrix} \begin{bmatrix} P_i'(\theta(t), V(t)) \\ Q_i'(\theta(t), V(t)) \end{bmatrix}$$
$$= \begin{bmatrix} \sum\limits_{j\in\mathcal{V}} \big[\gamma_{ij}(t)\sin(\theta_i(t) - \theta_j(t)) + \alpha_{ij}(t)\cos(\theta_i(t) - \theta_j(t))\big] \\ \sum\limits_{j\in\mathcal{V}} \big[-\gamma_{ij}(t)\cos(\theta_i(t) - \theta_j(t)) + \alpha_{ij}(t)\sin(\theta_i(t) - \theta_j(t))\big] \end{bmatrix}, \quad (3)$$

where $\psi = -\tan^{-1}(\zeta)$, $\alpha_{ij}(t) := V_i(t)V_j(t)(G_{ij} - B_{ij}\frac{1}{\zeta})\sin(\psi)$, $\gamma_{ij}(t) := V_i(t)V_j(t)(B_{ij}\sin(\psi) - G_{ij}\cos(\psi))$. In the new coordinates in (3), let $P_j^{(I)}(t)$ denote the active power generated by the inverter at node $j \in \mathcal{V}_p^{(I)}$, i.e.,

$$P_i^{(I)}(t) = P_i(\theta(t), V(t)), \quad (4)$$

and let $\ell_l(t)$ and $q_l(t)$ denote the active and reactive power demands at load $l \in \mathcal{V}_p^{(L)}$, i.e.,

$$\ell_l(t) = -P_l(\theta(t), V(t)), \;\; q_l(t) = -Q_l(\theta(t), V(t)). \quad (5)$$

In the remainder, we assume that after a change in the load, the proposed controller is fast enough to achieve its objectives before another load change occurs, thus, $\ell_l(t)$ and $q_l(t)$ are assumed to be constant, when the controller is being executed. Then, by differentiating (3) (along with (4)) for all $i \in \mathcal{V}$, we obtain that

$$\begin{bmatrix} \dot{P}^{(I)}(t) \\ \mathbf{0}_{n-m} \end{bmatrix} = L^{(p)}(\theta(t), V(t))\dot{\theta}(t) + K(\theta(t), V(t))\dot{V}^{(I)}(t) + N(\theta(t), V(t))\dot{V}^{(L)}(t), \quad (6)$$

$$\begin{bmatrix} \dot{Q}^{(I)}(t) \\ \mathbf{0}_{n-m} \end{bmatrix} = F(\theta(t), V(t))\dot{\theta}(t) + G(\theta(t), V(t))\dot{V}^{(I)}(t) + H(\theta(t), V(t))\dot{V}^{(L)}(t), \quad (7)$$

where $V^{(I)}(t) = [V_1(t), \ldots, V_m(t)]^{\mathrm{T}}$, $V^{(L)}(t) = [V_{m+1}(t), \ldots, V_n(t)]^{\mathrm{T}}$, $\mathbf{0}_{n-m}$ is the $(n-m)$-dimensional all-zeros vector, $K(\theta(t), V(t))$, $N(\theta(t), V(t))$, $F(\theta(t), V(t))$, $G(\theta(t), V(t))$, and $H(\theta(t), V(t))$ are defined accordingly, and $L^{(p)}(\theta(t), V(t)) \in \mathbb{R}^{n\times n}$ is defined as follows: $L_{ii}^{(p)}(\theta(t), V(t)) = -\sum\limits_{j\neq i, j\in\mathcal{V}} L_{ij}^{(p)}(\theta(t), V(t))$, $i \in \mathcal{V}$, and

$$L_{ij}^{(p)}(\theta(t), V(t)) = -\gamma_{ij}(t)\cos(\theta_i(t) - \theta_j(t)) + \alpha_{ij}(t)\sin(\theta_i(t) - \theta_j(t))), \;\; i \neq j.$$

Define a one-to-one map $\mathbb{I} : \mathcal{E} \to \mathbb{R}$ such that every e in the set $\{1, 2, \ldots, |\mathcal{E}|\}$ is arbitrarily assigned to exactly one edge $\{i, j\} \in \mathcal{E}$, i.e., $\mathbb{I}(\{i, j\}) = e$. We assign an arbitrary orientation to each edge $\{l, j\}$, and denote the set of all directed edges by $\vec{\mathcal{E}}$. Let (i, j) denote the edge oriented from i to j, and define a node-to-edge incidence matrix, M, as follows: for each $e = \mathbb{I}(\{i, j\})$, $M_{ie} = 1$ and $M_{je} = -1$, if $(i, j) \in \vec{\mathcal{E}}$, and, for all $e \in \{1, 2, \ldots, |\mathcal{E}|\}$, $M_{ie} = 0$ and $M_{je} = 0$, if $\{i, j\} \notin \mathcal{E}$. If $(i, j) \in \vec{\mathcal{E}}$, j is referred to as an out-neighbor of i, and i is referred to as an in-neighbor of j. Let $\mathcal{N}_i^+$ and $\mathcal{N}_i^-$ denote the sets of out-neighbors and in-neighbors of i. Let the Laplacian of $\mathcal{G}$ be denoted by $L(V(t)) \in \mathbb{R}^{n\times n}$ with $L_{ij}(V(t)) = -\gamma_{ij}(t)$, $i \neq j$, $i, j \in \mathcal{V}$, and $L_{ii}(V(t)) = \sum\limits_{l\neq i, l\in\mathcal{V}} \gamma_{il}(t)$, where a time-varying weight $\gamma_{ij}(t)$ is assigned to an edge $\{i, j\} \in \mathcal{E}$.

B. Control Objectives

Let $\omega_i(t) := \omega^* + \dot{\theta}_i(t)$ denote the local frequency at node i (we consider phases $\theta(t)$ in the reference frame rotating at the nominal frequency ω^*). The controller proposed for the system in (6) – (7) adjusts the inverter frequencies, $\omega_i(t)$, and voltage magnitudes, $V_i(t)$, $i \in \mathcal{V}^{(I)}$, to achieve the following control objectives:

- O1. track desired references, P_i^*, $i \in \mathcal{V}^{(I)}$, which balance the load, i.e., $\sum\limits_{i\in\mathcal{V}^{(I)}} P_i^* = \sum\limits_{l\in\mathcal{V}^{(L)}} \ell_l$;
- O2. maintain the frequency at each bus, $\omega_i(t)$, at the nominal value, ω^*; and
- O3. ensure that the voltage magnitudes at all buses are within some interval $\underline{V}_i \leq V_i(t) \leq \overline{V}_i$, $i \in \mathcal{V}$, where $\underline{V}_i$ and $\overline{V}_i$ are limits prescribed by operational considerations.

III. Frequency and Voltage Regulation: Homogeneous Inductance-to-Resistance Ratio

In this section, we propose frequency and voltage controllers for the case when the inductance-to-resistance ratio is homogeneous across all electrical lines, i.e., $\zeta = \frac{B_{ij}}{G_{ij}}$, $\forall \{i, j\} \in \mathcal{E}$.

A. Distributed Frequency Controller

For now, we fix voltage magnitudes at the inverter buses, i.e., $\dot{V}^{(I)}(t) = 0$, and assume that $N(\theta(t), V(t))\dot{V}^{(L)}(t) \approx 0$ (we will relax this later when we discuss voltage control), which allows us to simplify (6) as follows:

$$\begin{bmatrix} \dot{P}^{(I)}(t) \\ \mathbf{0}_{n-m} \end{bmatrix} = L^{(p)}(\theta(t), V(t))\dot{\theta}(t). \quad (8)$$

By partitioning $L^{(p)}(\theta(t))$ as follows

$$L^{(p)}(\theta(t)) = \begin{bmatrix} L_{II}^{(p)}(\theta(t)) & L_{IL}^{(p)}(\theta(t)) \\ L_{LI}^{(p)}(\theta(t)) & L_{LL}^{(p)}(\theta(t)) \end{bmatrix}, \quad (9)$$

where $L_{II}^{(p)}(\theta(t)) \in \mathbb{R}^{|\mathcal{V}^{(I)}|\times|\mathcal{V}^{(I)}|}$ and $L_{LL}^{(p)}(\theta(t)) \in \mathbb{R}^{|\mathcal{V}^{(L)}|\times|\mathcal{V}^{(L)}|}$, we can rewrite (8) equivalently as

$$\dot{P}^{(I)}(t) = S(\theta(t))\dot{\theta}^{(I)}(t), \quad (10)$$

$$\dot{\theta}^{(L)}(t) = -(L_{LL}^{(p)}(\theta(t)))^{-1}L_{LI}^{(p)}(\theta(t))\dot{\theta}^{(I)}(t), \quad (11)$$

where $\theta^{(I)}(t) = [\theta_1(t), \ldots, \theta_m(t)]^{\mathrm{T}}$, $\theta^{(L)}(t) = [\theta_{m+1}(t), \ldots, \theta_n(t)]^{\mathrm{T}}$, and $S(\theta(t))$ denotes the Schur complement of $L^{(p)}(\theta(t))$, i.e., $S(\theta(t)) = L_{II}^{(p)}(\theta(t)) - L_{IL}^{(p)}(\theta(t))(L_{LL}^{(p)}(\theta(t)))^{-1}L_{LI}^{(p)}(\theta(t))$. [The matrix $L_{LL}^{(p)}(\theta(t))$ is guaranteed to be invertible for all t since the frequency controller will be designed to enforce $|\theta_i(t) - \theta_j(t)| < \frac{\pi}{2}, \forall\{i,j\} \in \mathcal{E}$.] Note that $L^{(p)}(\theta(t))$ is the weighted Laplacian of the graph $\mathcal{G}$, and $S(\theta(t))$ is positive semidefinite with a single zero eigenvalue, the corresponding eigenvector of which being the m-dimensional all-ones vector, $\mathbb{1}_m$.

As shown in [7], the following frequency controller achieves objectives O1 and O2 for the model in (10) – (11):

$$\dot{\theta}_m(t) = 0, \quad (12)$$

$$\dot{\theta}_i(t) = -\alpha(P_i^{(I)}(t) - P_i^r(t)), \quad i = 1, \ldots, m-1, \quad (13)$$

$$P_i^r(t) = P_i^r(t_k^-) + \Delta P_i^r, \quad t_k \le t < t_{k+1}, \quad (14)$$

$$\Delta P_i^r = \frac{(P_i^* - P_i^{(I)}(t_0))(\delta - \underline{\delta}\sqrt{m})}{\left[(m+1)\sum_{i=1}^{m-1}(P_i^* - P_i^{(I)}(t_0))^2\right]^{\frac{1}{2}}}, \quad (15)$$

where t_0 is the time at which the frequency controller is triggered, $P^r(t_0^-) = P^{(I)}(t_0^-)$, with t_k denoting the time instant at which $P^r(t)$ is adjusted, which satisfies

$$t_k \ge \inf_t \arg_{t > t_{k-1}} \left\{ \max_{i<m} |P_i(t) - P_i^r(t)| \le \underline{\delta} \right\}, \quad (16)$$

where $\underline{\delta}$ is any positive constant less than $\frac{\delta}{\sqrt{m}}$, and δ is computed based on the $P_i^{(I)}(t_0)$'s and the P_i^*'s as shown in [7]. Here, we assume that $P_m^r(t) = \sum_{l\in\mathcal{V}^{(L)}} \ell_l - \sum_{i<m} P_i^r(t)$, which is known within a certain accuracy to inverter m.

The frequency controller is based on the idea of making the injections at the inverter buses follow a certain desired trajectory ending at the reference P^*; the injections follow this trajectory if they are guided by a time-varying reference $P^r(t)$ by using the controller in (13), which drives the injections to a piecewise-constant $P^r(t)$ if they are initially within a certain distance, δ, from $P^r(t)$. As described in (16), once the injections get close enough to $P^r(t)$ (as determined by the parameter $\underline{\delta}$), $P^r(t)$ is moved closer to P^* along the desired trajectory. The readjustment continues until $P^r(t)$ reaches P^*. The controller in (12) – (15) was proposed in [7] assuming fixed voltage magnitudes at all buses. Later, in this paper, we lift a portion of this assumption by proposing an event-triggered scheme for the voltage controller, but still maintaining the following assumption in the remainder.

Assumption 1.

$$(\dot{\theta}^{(I)}(t))^{\mathrm{T}}\dot{P}^{(I)}(t)\Big|_{\substack{\theta(t)\in\mathcal{S}(\frac{\pi}{2}) \\ \dot{V}^{(I)}(t)=0}} > 0, \text{ if } \mathbb{1}_m^{\mathrm{T}}\dot{\theta}^{(I)}(t) \neq 0, \quad (17)$$

where $\mathcal{S}(\frac{\pi}{2}) \coloneqq \{\theta \in \mathbb{R}^n : |\theta_i - \theta_j| < \frac{\pi}{2}, \forall\{i,j\} \in \mathcal{E}\}$.

Assumption 1 implies that when $\dot{V}^{(I)}(t) = 0$, the vector of injections at inverter buses, $P^{(I)}(t)$, mainly depends on $\theta(t)$, and the execution of the frequency controller leads to changes in the voltage magnitudes of the load buses electrically connected to the inverter buses, which do not substantially affect $P^{(I)}(t)$. It might seem difficult to check how reasonable this assumption is, but the often used assumption of having fixed voltage magnitudes at all buses and $\theta(t) \in \mathcal{S}(\frac{\pi}{2})$ (see, e.g., [7], [9], [10]), implies Assumption 1 since $S(\theta(t))$ is positive semidefinite and

$$(\dot{\theta}^{(I)}(t))^{\mathrm{T}}\dot{P}^{(I)}(t) = (\dot{\theta}^{(I)}(t))^{\mathrm{T}}S(\theta(t))\dot{\theta}^{(I)}(t) > 0,$$

if $\mathbb{1}_m^{\mathrm{T}}\dot{\theta}^{(I)}(t) \neq 0$. Additionally, for sufficiently small voltage phase angle differences between inverter nodes and their neighbors, (17) is satisfied if $\theta(t) \in \mathcal{S}(\frac{\pi}{2})$, which is a standard assumption in microgrid literature.

Let e_i denote a vector with all its entries equal to zero except for the i-th one, which is equal to 1. Let $R_{ij}(V)$ denote the effective resistance between nodes i and $j \in \mathcal{V}$ for a network with voltage magnitudes V_l, $l \in \mathcal{V}$; then, $R_{ij}(V) = (e_i - e_j)^{\mathrm{T}}[L(V)]^{\dagger}(e_i - e_j)$ [11, Definition 2.1]. The next result extends [7, Corollary 1] and shows that the frequency controller in (12) – (15) drives $P^{(I)}(t) \to P^*$ exponentially under Assumption 1; the proof is omitted.

Proposition 1. Consider the model in (6) – (7) with $\zeta = \frac{B_{ij}}{G_{ij}}$, $\forall\{i,j\} \in \mathcal{E}$. Suppose that $\dot{\theta}_m(t) = 0$, for all t, $\theta(t_0) \in \mathcal{S}(\frac{\pi}{2})$, and $P(t_0)$ and $\begin{bmatrix} P^* \\ -\ell \end{bmatrix}$ satisfy

$$\sup_{\underline{V}\le V\le\overline{V}} \|M^{\mathrm{T}}[L(V)]^{\dagger}P(t_0)\|_\infty \le \kappa\sin\epsilon, \quad (18)$$

$$\sup_{\underline{V}\le V\le\overline{V}} \|M^{\mathrm{T}}[L(V)]^{\dagger}\begin{bmatrix} P^* \\ -\ell \end{bmatrix}\|_\infty \le \kappa\sin\epsilon, \quad (19)$$

where $[L(V)]^{\dagger}$ is the pseudoinverse of $L(V)$. Let $\delta = \frac{1-\kappa}{\eta(\overline{V})}\sin\epsilon$, where $\epsilon \in (0, \frac{\pi}{2})$, $\kappa \in (0,1)$, and $\eta(V) = \max_{\{i,j\}\in\mathcal{E}} \frac{1.5m}{R_{ij}(V)}$ with m denoting the number of inverters. Then, under Assumption 1, the controller in (12) – (15) ensures that $P^{(I)}(t) \to P^*$ exponentially.

B. Distributed Voltage Controller

Now, we present a distributed event-triggered voltage controller that forces the voltage magnitudes at all buses to be within some limits prescribed by operational considerations.

First, note that it takes $\lceil |P_i(t_0) - P_i^*|/\Delta P_i^r \rceil$ steps to reach P_i^*, $i \in \mathcal{V}^{(I)}$, where $\lceil x \rceil$ is the smaller integer greater than or equal to x. Let $\mathcal{K} \subseteq \{1, 2, \ldots, \lceil |P_i(t_0) - P_i^*|/\Delta P_i^r \rceil\}$, and define a ball $\mathcal{B}^{(r)}(t) = \{x \in \mathbb{R}^m : \max_{i<m} |x_i - P_i^r(t)| \le \underline{\delta}\}$. The proposed voltage controller is event-triggered, and its

execution starts once $P^{(I)}(t) \in \mathcal{B}^{(r)}(t)$ at each $t = t_k$, $k \in \mathcal{K}$, and ends when $\underline{V}_i \le V_i(t) \le \overline{V}_i$, $i \in \mathcal{V}$. If the voltage controller pushes $P^{(I)}(t)$ out of the ball $\mathcal{B}^{(r)}(t)$ at $t = t_k^1$, i.e., $P^{(I)}(t_k^1) \notin \mathcal{B}^{(r)}(t_k^1)$, then, the execution of the voltage controller is paused until the frequency controller drives $P^{(I)}(t)$ inside the ball $\mathcal{B}^{(r)}(t)$, and resumed once $P^{(I)}(t) \in \mathcal{B}^{(r)}(t)$. Note that the frequency controller never stops its execution, and updates the reference $P^r(t)$ once the voltage controller completes its execution at $t = t_{k+1}$, when $\underline{V}_i \le V_i(t_{k+1}) \le \overline{V}_i$, $i \in \mathcal{V}$.

More formally, when $P^{(I)}(t) \in \mathcal{B}^{(r)}(t)$ at $t = t_k$, $k \in \mathcal{K}$, we fix $P^r(t) = P^r(t_k)$, $t \ge t_k$, and execute the voltage controller as follows. Let the power flows corresponding to $P(t)$ be denoted by $f_{ij}(t)$ in the transformed coordinates in (3), i.e., $f_{ij}(t) = \gamma_{ij}(t)\sin(\theta_i(t) - \theta_j(t))$, and $P_i(t) = \sum_{j\in\mathcal{V}} f_{ij}(t)$. Similarly, let $f^r_{ij}(t)$ denote the power flows corresponding to $P^r(t)$ in the transformed coordinates. Since there is only one unique $\theta \in \mathcal{S}(\frac{\pi}{2})$ corresponding to $P(t)$ (see [12, Corollary 1]) and the frequency controller ensures that $\theta(t) \in \mathcal{S}(\frac{\pi}{2})$, then, there is only one $f(t) = [f_{ij}(t)]^{\mathrm{T}}_{\{i,j\}\in\mathcal{E}}$ for any given $P(t)$ corresponding to $\theta(t)$. Since, by the definition of t_k in (16), $\max_{i<m} |P_i(t) - P^r_i(t)| \le \underline{\delta}$ for $t \ge t_k$, then, we have that $|f_{ij}(t) - f^r_{ij}(t_k)|$ is negligibly small for $\forall\{i,j\} \in \mathcal{E}$ and sufficiently small $\underline{\delta}$; thus,

$$\gamma_{ij}(t)\cos(\theta_i(t) - \theta_j(t)) = [\gamma^2_{ij}(t) - (f_{ij}(t))^2]^{\frac{1}{2}} \approx [\gamma^2_{ij}(t) - (f^r_{ij}(t_k))^2]^{\frac{1}{2}}, \quad (20)$$

which does not depend on $\theta(t)$. This implies that when the active power injections $P^{(I)}(t) \in \mathcal{B}^{(r)}(t)$, the reactive power injections are decoupled from the voltage phase angles, since by using (20) we can rewrite the reactive power injections in (3) as follows:

$$Q_i(t) = \sum_{j\in\mathcal{V}} \left(-\gamma_{ii}(t) - (\gamma^2_{ij}(t) - (f^r_{ij}(t_k))^2)^{\frac{1}{2}}\right). \quad (21)$$

To design the voltage controller, we linearize the model in (21) around $V(t) = V(t_k)$:

$$\Delta Q(t) = A(V(t_k), \gamma(t_k), f^r(t_k))\Delta V(t), \quad (22)$$

where $\Delta Q(t) = [Q_1(t) - Q_1(t_k), \dots, Q_n(t) - Q_n(t_k)]^{\mathrm{T}}$, $\Delta V(t) = [V_1(t) - V_1(t_k), \dots, V_n(t) - V_n(t_k)]^{\mathrm{T}}$, $\gamma(t) = [\gamma_{ij}(t)]^{\mathrm{T}}_{\{i,j\}\in\mathcal{E}}$, and $A(\gamma(t_k), f^r(t_k)) = [A_{ij}(\gamma(t_k), f^r(t_k))]_{\{i,j\}\in\mathcal{E}} \in \mathbb{R}^{n\times n}$ with

$$A_{ij}(\gamma(t_k), f^r(t_k)) = -\frac{\gamma^2_{ij}(t_k)}{V_j(t_k)(\gamma^2_{ij}(t_k) - (f^r_{ij}(t_k))^2)^{\frac{1}{2}}},$$
$$A_{ii}(V(t_k), f^r(t_k)) = -2\gamma_{ii}(t_k)/V_i(t_k) - \sum_{l\in\mathcal{V}} \frac{\gamma^2_{il}(t_k)}{V_i(t_k)(\gamma^2_{il}(t_k) - (f^r_{il}(t_k))^2)^{\frac{1}{2}}}.$$

Let m_i denote the droop coefficient at inverter i for reactive power sharing and $\underline{Q}_i$ and $\overline{Q}_i$ the minimum and maximum reactive power capacity at inverter $i \in \mathcal{V}^{(I)}$. We achieve the voltage regulation Objective O3 by solving the following quadratic optimization problem at each t_k, $k \in \mathcal{K}$:

$$\min_{\Delta q, \Delta v \in \mathbb{R}^n} \frac{1}{2}\|\Delta q - A(V(t_k), \gamma(t_k), f^r(t_k))\Delta v\|_2^2 + \underbrace{\frac{\beta}{2}\Delta q^{\mathrm{T}} W L(V(t_k)) W \Delta q}_{\text{reactive power sharing}} \quad (23)$$

$$\text{subject to } \Delta\underline{Q}_i(t_k) \le \Delta q_i \le \Delta\overline{Q}_i(t_k), \quad (24)$$
$$\underbrace{\Delta\underline{V}_i(t_k) \le \Delta v_i \le \Delta\overline{V}_i(t_k)}_{\text{voltage regulation}}, i \in \mathcal{V}, \quad (25)$$

which we refer to as QP1, where $\Delta\overline{V}_i(t_k) = \overline{V}_i - V(t_k)$, $\Delta\underline{V}_i(t_k) = \underline{V}_i - V(t_k)$, $i \in \mathcal{V}$, $\Delta\underline{Q}_i(t_k) = \underline{Q}_i - Q_i(t_k)$, $\Delta\overline{Q}_i(t_k) = \overline{Q}_i - Q_i(t_k)$, $i \in \mathcal{V}^{(I)}$, $\Delta\underline{Q}_l(t_k) = \Delta\overline{Q}_l(t_k) = 0$, $l \in \mathcal{V}^{(L)}$, $\beta \ge 0$ is a constant parameter, and $W_{ii} = m_i^{-1}$. When $\beta = 0$, QP1 purely focuses on achieving O3; by increasing β, reactive power sharing can be recovered to a certain extent. Since QP1 is ill-conditioned, to find its solution we use a scaled version of the gradient projection method (see, e.g., [13, Section 2.3]), the iterations of which are given by

$$\Delta q(t) = \big[\Delta q(t-h) - sD(\beta W L W \Delta q(t-h) + \Delta q(t-h) - A(V(t_k), \gamma(t_k), f^r(t_k))\Delta v(t-h))\big]^+, \quad (26)$$
$$\Delta v(t) = \big[\Delta v(t-h) - sA^{\mathrm{T}}(V(t_k), \gamma(t_k), f^r(t_k))(\Delta q(t-h) - A(V(t_k), \gamma(t_k), f^r(t_k))\Delta v(t-h))\big]^+, \quad (27)$$

where D is a diagonal matrix with strictly positive diagonal entries, $D_{ii} = V_i(t_k)$, s is a constant step-size, h is the iteration index, $\Delta q(t) = [\Delta q_1(t), \dots, \Delta q_n(t)]^{\mathrm{T}}$, and $\Delta v(t) = [\Delta v_1(t), \dots, \Delta v_n(t)]^{\mathrm{T}}$, with local variables $\Delta q_i(t)$ and $\Delta v_i(t)$ updated at inverter $i \in \mathcal{V}^{(I)}$, and $\Delta v_l(t)$ updated at load $l \in \mathcal{V}^{(L)}$. The iterations in (26) – (27) are executed on-line across all nodes, and each inverter i adjusts its voltage magnitude by $\Delta v_i(t)$ at time t as follows:

$$V_i(t) = V_i(t_k) + \Delta v_i(t), \quad t_k \le t < t_{k+1}. \quad (28)$$

The system in (6) – (7) and (28) is considered to be asymptotically stable if $P^{(I)}(t) \to P^*$ and $V(t) \to V^*$, where $\underline{V}_i \le V^*_i \le \overline{V}_i$, $i \in \mathcal{V}$. The result in the proposition that follows establishes that the system in (6) – (7) and (28) is asymptotically stable under the proposed frequency and voltage controllers in (12) – (15) and (28), the proof of which is omitted.

Proposition 2. Let Assumption 1 hold, $\zeta = \frac{B_{ij}}{G_{ij}}$, $\forall\{i,j\} \in \mathcal{E}$, and $\beta = 0$. Then, the closed loop system in (6) – (7), (12) – (15), and (28) is asymptotically stable.

Remark 1. Intuitively, the frequency controller keeps the power flows on manifold

$$\mathcal{M}^r(t) := \big\{(\theta, V) \in \mathbb{R}^n \times \mathbb{R}^n : f_{ij}(\theta, V) = f^r_{ij}(t), \{i,j\} \in \mathcal{E}\big\}, \quad (29)$$

and the voltage controller in (28) achieves voltage regulation by moving along this manifold. If the voltage controller pushes $(\theta(t), V(t))$ away from the manifold $\mathcal{M}^r(t)$, then, the frequency controller drives $(\theta(t), V(t))$ back to $\mathcal{M}^r(t)$. Our approach is different from the controllers based on a mere linearization of the system in (6) – (7) around an equilibrium point (θ^*, V^*) in many ways because (i) the model for the reactive power injections in (22) is independent of $\theta(t)$ and, in fact, it is a linearization around $\mathcal{M}^r(t)$ and not around a particular point (θ^*, V^*), (ii) asymptotic stability is with respect to $\mathcal{M}^r(t)$ and not with respect to a particular point (θ^*, V^*), i.e., the proposed controllers drive $(\theta(t), V(t))$ to $\mathcal{M}^r(t)$; thus, ensuring convergence to $\mathcal{M}^r(t)$ establishes that Objectives O1-O3 are met. ■

IV. Voltage Regulation: Non-Homogeneous Inductance-to-Resistance Ratio

In this section, we extend the voltage controller introduced earlier to the general case when the inductance-to-resistance ratio is not necessarily homogeneous across electrical lines. As in the case of the homogeneous inductance-to-resistance ratio case, we assume that there is a frequency controller, which maintains the power flows on a certain manifold, and the voltage controller in (28) achieves voltage regulation by moving along this manifold.

A. Voltage Controller

Similar to the execution of the voltage controller for homogeneous electrical lines introduced in Section III, when $\max_{i<m} |P_i(t) - P_i^r(t)|$ is sufficiently small, we activate and execute the voltage controller as discussed next. Here, $P^r(t_k)$ can be a piecewise-constant or fixed reference depending on the type of the frequency controller. Define $\rho_{ij}(t) := (\gamma_{ij}(t)^2 + \alpha_{ij}(t)^2)^{\frac{1}{2}}$, $\delta_{ij}(t) := \arccos\left(\frac{\gamma_{ij}(t)}{(\gamma_{ij}^2(t)+\alpha_{ij}^2(t))^{\frac{1}{2}}}\right)$, and $\phi_{ij}(t) := \rho_{ij}(t)\sin(\theta_i(t) - \theta_j(t) + \delta_{ij}(t))$; then,

$$\phi_{ij}(t) = \gamma_{ij}(t)\sin(\theta_i(t) - \theta_j(t)) + \alpha_{ij}(t)\cos(\theta_i(t) - \theta_j(t)),$$

and $P_i(t) = \sum_{j\in\mathcal{V}} \phi_{ij}(t)$, i.e., $\phi_{ij}(t)$'s are the power flows corresponding to $P(t)$. Similarly, let $\phi_{ij}^r(t)$'s denote the power flows corresponding to $P^r(t)$. For tree networks, if $P^{(I)}(t) \in \mathcal{B}^{(r)}(t)$, then, it is easy to show that $|\phi_{ij}(t) - \phi_{ij}^r(t_k)|$, $\forall\{i,j\} \in \mathcal{E}$, are negligibly small. For cyclic networks, we assume that $|\phi_{ij}(t) - \phi_{ij}^r(t_k)|$ are negligibly small even when $\{i,j\}$ lies on a cycle. As in the case of homogeneous electrical lines, if we enforce that $P^{(I)}(t) \in \mathcal{B}^{(r)}(t)$, we can decouple reactive power injections from voltage phase angles, since we can rewrite the expression for the reactive power injections in (3) as follows:

$$Q_i(t) = \sum_{j\in\mathcal{V}} \left(-\gamma_{ii}(t) - (\rho_{ij}^2(t) - \phi_{ij}^2(t_k))^{\frac{1}{2}}\right), \qquad (30)$$

which is similar to (21). The same optimization problem OP1 and iterations in (26) – (27) can be used to adjust the voltage magnitude reference in (28) by replacing $A(V(t_k), \gamma(t_k), f^r(t_k))$ with $A(V(t_k), \rho(t_k), \phi^r(t_k))$, where $\rho(t) = [\rho_{ij}(t)]^{\mathrm{T}}_{\{i,j\}\in\mathcal{E}}$ and $\phi^r(t) = [\phi_{ij}^r(t)]^{\mathrm{T}}_{\{i,j\}\in\mathcal{E}}$. The stability analysis and results can be also extended; for example, one can show the existence of δ, which ensures that the frequency controller in (12) – (15) makes the injections $P^{(I)}(t)$ converge to P^* for the case when the inductance-to-resistance ratios of the electrical lines are spread over a certain bounded interval. Additionally, under an assumption similar to Assumption 1, it can be shown that the frequency controller in (12) – (15) can drive and maintain power flows on manifold $\mathcal{M}^r(t) = \{(\theta, V) \in \mathbb{R}^n \times \mathbb{R}^n : \phi_{ij}(\theta, V) = \phi_{ij}^r(t), \{i,j\} \in \mathcal{E}\}$. For simplicity, we assume that there exists a certain frequency controller which drives and maintains power flows on manifold $\mathcal{M}^r(t)$. Now, we state a stability result, which is a trivial extension of Proposition 2.

Proposition 3. Suppose $(\theta(t), V(t))$ is sufficiently close to $\mathcal{M}^r(t)$ for all t. If $\beta = 0$, then, under the voltage controller in (28), $V(t)$ converges asymptotically to V^*, which satisfies the box constraints, $\underline{V}_i \le V_i^* \le \overline{V}_i$, $i \in \mathcal{V}$.

B. Distributed Flow Computation

The voltage controller in (28) requires the knowledge of the power flows in the transformed coordinates in (3), $\phi^r(t_k)$, for non-homogenous inductance-to-resistance ratio case (homogeneous case is a special case), which can be computed distributively by using an algorithm similar to the so-called feasible flow algorithm (see, e.g., [10]) as discussed next.

To find the power flows corresponding to P^* for electrical lines with non-homogeneous inductance-to-resistance ratios, we solve the following network flow problem

$$\begin{aligned} &\text{find } \phi \in \mathbb{R}^{2|\mathcal{E}|} \\ &\text{subject to } P_i^* = \sum_{j\in\mathcal{V}} \phi_{ij}, && (31) \\ &\underline{\phi}_{ij} \le \phi_{ij} \le \overline{\phi}_{ij},\ \{i,j\} \in \mathcal{E} && (32) \\ &\phi_{ji} = g_{ij}(\phi_{ij}), && (33) \end{aligned}$$

which we refer to as NFP, where $\phi_e = \phi_{ij}$ and $\phi_{|\mathcal{E}|+e} = \phi_{ji}$, $e = \mathbb{I}(\{i,j\})$, $(i,j) \in \vec{\mathcal{E}}$, $\overline{\phi}_{ij} := \gamma_{ij}\sin(\epsilon_{ij}) + \alpha_{ij}\cos(\epsilon_{ij})$, $\underline{\phi}_{ij} := -\gamma_{ij}\sin(\epsilon_{ij}) + \alpha_{ij}\cos(\epsilon_{ij})$, with $\epsilon_{ij} := \tan^{-1}\left(\frac{\gamma_{ij}(t)}{\alpha_{ij}(t)}\right)$, and $g_{ij}(\cdot)$ is a monotonically non-increasing convex function; satisfying (32) ensures that $|\theta_i - \theta_j| < \epsilon_{ij}$, which guarantees the existence of $g_{ij}(\cdot)$ [14]. Note that for homogeneous inductance-to-resistance ratio case, $g_{ij}(\cdot) \equiv -1$. NFP can be cast as the following optimization problem:

$$\min_{\phi\in\mathbb{R}^{2|\mathcal{E}|}} \|b\|_2^2,\ \phi \text{ subject to constraints in (31) – (33)},$$

which we refer to as OP, where $b = [b_1, \ldots, b_n]^{\mathrm{T}}$, with $b_i := P_i^* - \sum_{j\in\mathcal{N}_i} \phi_{ij}$ denoting the flow balance at node i. Because of the nonlinear constraint in (33), the constraint set for ϕ is non-convex. Before we proceed with an approach to solve OP, we assume that for cyclic $\mathcal{G}$, there is at least one node j in each cycle $\mathcal{C}_i$ that can measure flow along an out-going edge on cycle $\mathcal{C}_i$. Recall that $\mathcal{E}^{(c)}$ is the set of all edges lying on cycles along which flows are measured, and $\mathcal{G}' = \{\mathcal{V}, \mathcal{E}\setminus\mathcal{E}^{(c)}\}$, which is, clearly, a tree. Let $\vec{\mathcal{E}}^{(c)}$ denote the set of directed edges, which result from giving an orientation to the edges of $\mathcal{E}^{(c)}$; let each node $i \in \mathcal{V}$ execute the following gradient-type iterations to estimate the flows along the outgoing edges, $(i,j) \in \vec{\mathcal{E}} \setminus \vec{\mathcal{E}}^{(c)}$:

$$\phi_{ij}[k+1] = \left[\phi_{ij}[k] + s(b_i[k] - b_j[k])\right]_{-\rho_{ij}}^{\rho_{ij}}, \tag{34}$$

where $s > 0$ is a constant step-size. Let ϕ^* denote the solution of OP. Then, the power flows along the edges in $\vec{\mathcal{E}}^{(c)}$ are set to the measured values:

$$\phi_{ij}[k+1] = \phi_{ij}^*, \tag{35}$$

where ϕ_{ij}^* is assumed to be equal to the measured flow along edge $(i,j) \in \vec{\mathcal{E}}^{(c)}$. The estimates of the flows along the incoming edges, $(j,i) \in \vec{\mathcal{E}} \setminus \vec{\mathcal{E}}^{(c)}$, are transmitted to node i which, then, updates

$$\phi_{ij}[k+1] = g_{ji}(\phi_{ji}[k+1]), \tag{36}$$

according to the constraint in (33). Now, we state, without proof, the convergence result for the proposed flow computation scheme.

Proposition 4. Consider the network flow model in (31) – (33) and suppose ϕ_{ij}^*, $\{i,j\} \in \mathcal{E}^{(c)}$, are known from measurements. Then, the iterations in (34) converge to ϕ_{ij}^*, $(i,j) \in \vec{\mathcal{E}} \setminus \vec{\mathcal{E}}^{(c)}$, which satisfy the constraints in (31) – (33).

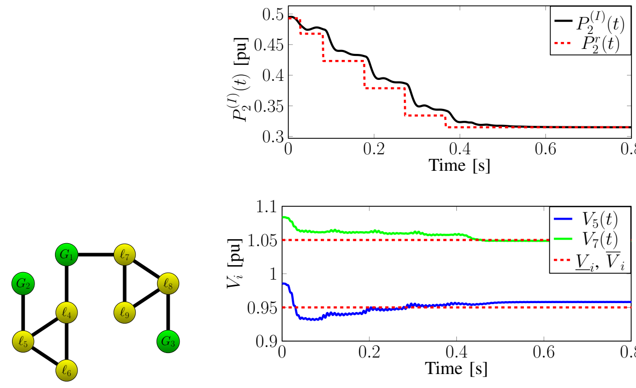
Clearly, the proposed iterative scheme is distributed since each node i requires only balances of its out-neighbors, $j \in \mathcal{N}_i^+$, and the flows along the in-coming edges from its in-neighbors, $j \in \mathcal{N}_i^-$.

V. Numerical Example

In this section, we present numerical results for a microgrid with its network topology as described in Fig. 1a. Here, nodes 1, 2 and 3 correspond to generators, with the remaining nodes corresponding to loads. In the example, we set $\rho = 0$, $\underline{V}_i = 0.95$ pu and $\overline{V}_i = 1.05$ pu, $i \in \mathcal{V}$, and the inductance-to-resistance ratio varies across the electrical lines within an interval $[1, 3]$. In Fig. 1b, because of the load perturbation at time $\tau = 0.05$ s, $V_5(\tau) < \underline{V}_5$ and $V_7(\tau) > \overline{V}_7$, which triggers the voltage controller in (28) at each inverter node, and each node $i \in \mathcal{V}$ updates its local variables $\Delta q_i(t)$ and $\Delta v_i(t)$ according to the (26) – (27). Triggering occurs automatically since once $V_i(t)$ violates its operational limits, it triggers and perturbs the iterations in (26) – (27). After a certain time, the voltage controller is able to push the voltage magnitudes at buses 5 and 7 back within their operational limits. Note that in Fig. 1b, the frequency controller in (12) – (15) is able to drive the active power injections to the desired references while the voltage controller is being executed. Also, as discussed previously, the voltage controller is executed only when the active power injections are close enough to the piecewise-constant reference $P^r(t)$.

VI. Concluding Remarks

In this paper, we proposed distributed controllers for frequency and voltage regulation in islanded ac microgrids executed via an event-triggered scheme. The frequency controller keeps the power flows on a certain manifold to achieve desired active power sharing, and the voltage controller achieves voltage regulation by moving along this manifold. Additional parameter in the voltage controller allows to recover reactive power sharing within a certain accuracy.



(a) (b)

Fig. 1: (a) 9-bus microgrid topology, (b) evolution of $P_2^{(I)}(t)$, $V_5(t)$, and $V_7(t)$.

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