Convex Relaxations of the Network Flow Problem Under Cycle Constraints

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***Abstract*—In this paper, we consider the problem of optimizing the flows in a lossless flow network with additional non-convex cycle constraints on nodal variables; such constraints appear in several applications, including electric power, water distribution and natural gas networks. This problem is a non-convex version of the minimum-cost network flow problem (NFP), and to solve it, we propose three different approaches. One approach is based on solving a convex approximation of the problem, obtained by augmenting the cost function with an entropy-like term to relax the non-convex constraints. We show that the approximation error, for which we give an upper bound, can be made small enough for practical use. An alternative approach is to solve the classical NFP, i.e., without the cycle constraints, and solve a separate optimization problem afterwards in order to recover the actual flows satisfying the cycle constraints; the solution of this separate problem maximizes the individual entropies of the cycles. The third approach is based on replacing the non-convex constraint set with a convex inner approximation, which yields a suboptimal solution for the cyclic networks with each edge belonging to at most two cycles. We validate the practical usefulness of the theoretical results through numerical examples, in which we study the standard test systems for water and electric power distribution networks.**

I. Introduction

We consider a network of nodes that are interconnected via links. Each node represents either a supplier or a consumer of a certain commodity, and links allow this commodity to flow between nodes, where the amount of flow is restricted by the capacity of each link. Assuming that links are lossless, the difference of commodity out-flows and in-flows at each node needs to be equal to the amount of commodity injected (positive if the corresponding node is a supplier, and negative otherwise). A typical objective here is to minimize a certain cost function, the arguments of which are the link flows, and possibly the nodal commodity injections; this problem is referred to as the minimum-cost network flow problem (see, e.g., [1]–[4]). In this paper, in addition to the aforementioned constraints in the classical formulation, we assume that the flow of the commodity along every link is a nonlinear function of the difference of certain nodal variables associated with the link end nodes. This relation between flows and nodal variables further imposes some equality constraints on the flows along every cycle in the network.

The setting in this paper appears in several engineering applications, including electric power systems (see, e.g., [5], [6]), water supply systems (see, e.g., [7]–[12]), and natural gas networks [13]. In water supply systems, the problem can be viewed as a pressure head assignment problem, where the objective is to supply enough pressure to deliver water to all parts of the water network; pressure heads are identified as the nodal variables (see, e.g., [7], [9], [11], [12]). In electric power systems, the optimal generation dispatch problem can be formulated as a network flow problem with the voltage phase angles identified as the nodal variables governing the power flows along the electrical lines. The so-called generalized network flow problem (GNF) has been studied in [6] with application to electric power systems with lossy lines; the authors of [6] propose a convexification method for the GNF, which is exact regardless of the topology of the network. However, the authors of [6] do not consider the same cycle constraints imposed by the nodal variables that we consider in this paper.

In the literature, the classical network flow problem is a convex optimization problem (see, e.g., [1]–[4]). Network flow problems with non-convex cost function have been considered in [14]–[17], and tackled by using branch-and-bound, dynamic programming techniques and local search. The aforementioned additional cycle constraints on the flows that we consider in our problem formulation are non-convex; we address this non-convexity using three approaches.

In the first approach, we define an entropy-like function for cycles in the network, and add this function as a penalty term to the cost function; this guarantees a satisfaction of the cycle constraints at the optimal solution. However, this penalty-based approach introduces a certain approximation of the original cost function; we show that this approximation can be made as accurate as possible by decreasing the penalty weight. The idea of relaxing the non-convex constraints using entropy-like penalty functions can also be found in several computer vision applications and combinatorial problems (see, e.g., [18], [19]), as well as in matrix scaling problems [20]. In the second approach, we solve a modified version of the original problem, where we remove the cycle constraints, and in order to obtain the actual flows satisfying the cycle constraints, we solve a separate optimization problem, the solution of which maximizes the value of the individual entropy-like functions associated to each link in a cycle. In our previous work [21], we proposed methods, which are very similar to these two approaches, to solve the NFP with focus on power systems and for the networks with edge-disjoint cycles. However, the previous work does not provide the proofs of the main results. In this paper, we also pursue a more general setting that enables them to be used in different engineering applications and for general networks. We propose an additional approach, in which we convexify the original problem by replacing the

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non-convex constraint set with its convex inner approximation. This approach builds on the results in [22], where the authors determine a convex inner approximation of the non-convex constraint set for the networks with edge-disjoint cycles, and solve the convexified problem that results.

The first approach is guaranteed to converge to the solution that satisfies the cycle constraints only if the flows on the lines are not binding; for networks with many congested lines, the approach does not guarantee convergence to such solution. The second approach can be used to estimate the actual flows that satisfy the cycle constraints for a given set of injections. The third approach does not work well for congested networks since the inner approximation, which roughly relies on lowering the line capacities along cyclic paths, might become infeasible.

II. Preliminaries

In this section, we first introduce a few graph-theoretic notions that are relevant to our work; the reader is referred to [23] for a more in-depth discussion on the topic. Then, we formulate the network flow problem under cycle constraints, and provide some discussion of where this problem arises in practice.

A. Graph-Theoretic Notions

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote an undirected graph, where $\mathcal{V} := \{1, 2, \ldots, n\}$ is the set of nodes, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, with $\{l, j\} \in \mathcal{E}$ if nodes l and j are connected. Let γ_{lj} denote some positive weight assigned to edge $\{l, j\} \in \mathcal{E}$, i.e., $\gamma_{lj} > 0$ if $\{l, j\} \in \mathcal{E}$, and zero, otherwise. Let $\mathcal{N}(i)$ and δ_i respectively denote the set of neighbors and the degree of node i, i.e., $\mathcal{N}(i) = \{j \in \mathcal{V} : \{i, j\} \in \mathcal{E}\}$, and $\delta_i = |\mathcal{N}(i)|$.

Define a one-to-one map, $\mathbb{I} : \mathcal{E} \to \mathbb{R}$, such that every e in the set $\{1, 2, \ldots, |\mathcal{E}|\}$ is arbitrarily assigned to exactly one edge $\{l, j\} \in \mathcal{E}$, i.e., $\mathbb{I}(\{l, j\}) = e$. We also arbitrarily assign an orientation to each edge in $\mathcal{E}$, and denote by (l, j) the edge oriented from l to j, where l is the start node and j is the end node of the edge. Let $\vec{\mathcal{E}}$ denote the set of oriented edges that result from this arbitrarily chosen orientation (note that $|\vec{\mathcal{E}}| = |\mathcal{E}| =: \ell$); then, based on this orientation, we can define a node-to-edge incidence matrix, $M \in \mathbb{R}^{n \times \ell}$, as follows. For each e, $M_{ie} = 1$ and $M_{je} = -1$, if $e = \mathbb{I}(\{i, j\})$ and $(i, j) \in \vec{\mathcal{E}}$, and $M_{ie} = 0$ and $M_{je} = 0$, if $e \neq \mathbb{I}(\{i, j\})$.

Let $\mathcal{T}$ denote a spanning tree in $\mathcal{G}$; then, an edge of $\mathcal{G}$ that does not belong to $\mathcal{T}$, and the path in $\mathcal{T}$ between the vertices of this edge form a cycle, referred to as an undirected *fundamental cycle* [23, Definition 2-8]. Since there are $\ell - n + 1$ edges which do not belong to $\mathcal{T}$, there exist $c := \ell - n + 1$ fundamental cycles in $\mathcal{G}$.

Let $\mathcal{C}_i = (\mathcal{V}_i, \mathcal{E}_i)$, $i = 1, \ldots, c$, denote an undirected fundamental cycle in $\mathcal{G}$, formed with respect to $\mathcal{T}$, where the vertex set $\mathcal{V}_i = \{i_1, i_2, \ldots, i_{d_i}\} \subseteq \mathcal{V}$, $d_i = |\mathcal{V}_i|$, and the edge set $\mathcal{E}_i = \{\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_{d_i}, i_1\}\} \subseteq \mathcal{E}$. The oriented counterpart of $\mathcal{C}_i$ is denoted by $\vec{\mathcal{C}_i}$ with the orientation of the edges being determined by traversing the cycle in one direction, i.e., either clockwise or counterclockwise, and the directed edge set defined as $\vec{\mathcal{E}_i}$. Additionally, we can define the entries of the fundamental cycle matrix, denoted by $N \in \mathbb{R}^{c \times \ell}$, as follows:

$$N_{ie} = \begin{cases} 1 & \text{if } (j, l) \in \vec{\mathcal{E}_i}, M_{je} = 1, \\ -1 & \text{if } (j, l) \in \vec{\mathcal{E}_i}, M_{je} = -1, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

where $e = \mathbb{I}(\{j, l\})$; then, we have the following result (see, e.g., [23, Theorem 4-6]):

$$MN^\top = \mathbf{0}. \tag{2}$$

B. Network Flow Problem Under Cycle Constraints

Consider a network of nodes, the topology of which is described by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with some arbitrarily chosen orientation for the edges in $\mathcal{E}$ inducing $\vec{\mathcal{E}}$. Assume the graph contains c oriented fundamental cycles, $\vec{\mathcal{C}}_1, \vec{\mathcal{C}}_2, \ldots, \vec{\mathcal{C}}_c$, formed with respect to a spanning tree, $\mathcal{T}$, of $\mathcal{G}$, and orientation as determined by that of $\vec{\mathcal{E}}$. Let u_i denote the *injection* of a commodity into the network via node $i \in \mathcal{V}$, and let f_{lj} denote the *flow* of this commodity along edge $(l, j) \in \vec{\mathcal{E}}$; the injection u_i and the flows incident to node i are related through the flow balance equation given below

$$u_i = \sum_{(i,j) \in \vec{\mathcal{E}}} f_{ij} - \sum_{(l,i) \in \vec{\mathcal{E}}} f_{li}.$$

The relation between the nodal injections and edge flows can be compactly written as

$$u = Mf, \tag{3}$$

where $u = [u_1, \ldots, u_n]^\top$, $f := [\{f_{lj}\}_{(l,j) \in \vec{\mathcal{E}}}]^\top$, where with a slight abuse of notation, $f_e := f_{ij}$, $e = \mathbb{I}(\{i, j\})$. For a given u, it follows from (2) that any two solutions of (3), $f^{(1)}$ and $f^{(2)}$, can be related as follows:

$$f^{(1)} = f^{(2)} + N^\top \mu, \tag{4}$$

where $\mu \in \mathbb{R}^c$.

In the adopted network flow model, injections and flows are constrained to lie within some upper and lower bounds: $\underline{u}_i \leq u_i \leq \overline{u}_i$, $i \in \mathcal{V}$, and $\underline{f}_{lj} \leq f_{lj} \leq \overline{f}_{lj}$, $(l, j) \in \vec{\mathcal{E}}$, where $\underline{f}_{lj} = -\overline{f}_{lj}$ and $\overline{f}_{lj} > 0$. In addition, we assume that the edge flows depend on certain nodal variables, denoted by θ_i, $i \in \mathcal{V}$, via a functional relation of the form

$$f_{lj} = \gamma_{lj} g(\theta_l - \theta_j),$$

where $g(\cdot) : \mathcal{B} \to \mathcal{Y}$ is a continuous and odd function, i.e., $g(x) = -g(-x)$, with $\mathcal{B} = \{x \in \mathbb{R} : -\phi \leq x \leq \phi\}$, and $\mathcal{Y} = \{y \in \mathbb{R} : -g(\phi) \leq y \leq g(\phi)\}$, for some constant $\phi \in \mathbb{R}$. The relation between the nodal variables and the edge flows can be compactly written as

$$f = \Gamma g(M^\top \theta), \tag{5}$$

where $\Gamma := \operatorname{diag}(\gamma)$, $\gamma_e := \gamma_{lj}$, $e = \mathbb{I}(\{l, j\})$, and $g(M^\top \theta) = [\{g(\theta_l - \theta_j)\}_{(l,j) \in \vec{\mathcal{E}}}]^\top \in \mathbb{R}^\ell$. Define $h := g^{-1} : \mathcal{Y} \to \mathcal{B}$, then,

$$\theta_l - \theta_j = h\left(\frac{f_{lj}}{\gamma_{lj}}\right).$$

We assume that $h(\cdot)$ is monotonically increasing, and $g(\cdot)$ is bijective over $\mathcal{B}$; then, it follows that $h(\cdot)$ is continuous over $\mathcal{Y}$ (see, e.g., [24, Theorem 4.17]). Here it is important to note that the assumptions on $g(\cdot)$ introduced above are consistent with the physics of the applications discussed in Section II-D.

C. Network Flow Problem Formulation

We are interested in finding the injections and nodal variables that solve the following problem, which we refer to as P0:

$$\mathbf{P}0: \min_{u,\theta\in\mathbb{R}^n} \sum_{i\in\mathcal{V}} F_i(u_i) \tag{6}$$

$$\text{subject to } u = M\Gamma g(M^\top\theta), \tag{7}$$

$$\underline{u} \le u \le \overline{u}, \tag{8}$$

$$\|M^\top\theta\|_\infty \le \phi, \tag{9}$$

where $F_i(\cdot)$ is a convex, strictly increasing and continuously differentiable cost function, $\|M^\top\theta\|_\infty = \max_{(i,j)\in\vec{\mathcal{E}}} |\theta_i - \theta_j|$; this problem is hard to solve because of the non-convex constraint in (7).

Instead of directly solving P0 in terms of the injections and nodal variables, we introduce the following problem, formulated in terms of the injections and flows:

$$\mathbf{P}1: \min_{u\in\mathbb{R}^n, f\in\mathbb{R}^\ell} \sum_{i\in\mathcal{V}} F_i(u_i) \tag{10}$$

$$\text{subject to } u = Mf, \tag{11}$$

$$\underline{u} \le u \le \overline{u}, \tag{12}$$

$$\underline{f} \le f \le \overline{f}, \tag{13}$$

$$Nh(\Gamma^{-1}f) = 0, \tag{14}$$

where $h(\Gamma^{-1}f) = [\{h(f_{lj}/\gamma_{lj})\}_{(l,j)\in\vec{\mathcal{E}}}]^\top \in \mathbb{R}^\ell$,

$$\overline{f} = [\{\overline{f}_{lj}\}_{(l,j)\in\vec{\mathcal{E}}}]^\top \in \mathbb{R}^\ell, \;\; \underline{f} = [\{\underline{f}_{lj}\}_{(l,j)\in\vec{\mathcal{E}}}]^\top \in \mathbb{R}^\ell,$$

with $\overline{f}_{lj} = -\underline{f}_{lj} := \gamma_{lj} g(\phi)$. The cycle constraint in (14) ensures that the nodal variables, that correspond to the flows, add up to zero along each cycle, i.e.,

$$\sum_{(l,j)\in\vec{\mathcal{E}}_i} (\theta_l - \theta_j) = 0, \quad \forall \mathcal{C}_i \in \mathcal{G}.$$

Remark 1. Then, the cycle constraint in (14) can be rewritten explicitly for each cycle $\mathcal{C}_i$ as follows:

$$\sum_{e=1}^{\ell} n_e^{(i)} h\left(\frac{f_e}{\gamma_e}\right) = 0, \tag{15}$$

where $n^{(i)}$ denotes the i-th column of $N^\top$. Based on the orientation of $\vec{\mathcal{C}}_i$, and

$$-f_{lj} = -\gamma_{lj} g(\theta_l - \theta_j) = \gamma_{lj} g(\theta_j - \theta_l) =: f_{jl}, \;\; (l,j) \in \vec{\mathcal{E}}, \tag{16}$$

an alternative expression for (15) is given by

$$\sum_{(l,j)\in\vec{\mathcal{E}}_i} h\left(\frac{f_{lj}}{\gamma_{lj}}\right) = 0. \tag{17}$$

In the remainder, we use both (15) and (17) as appropriate, and when left unspecified as in (15), the orientation of flow f_{lj} is always assumed to be consistent with the chosen orientation, $(l,j) \in \vec{\mathcal{E}}$. ■

Let $\mathcal{Q}$ and $\mathcal{S}$ denote the constraint sets of P0 and P1, respectively, i.e., $\mathcal{Q} := \{(u,\theta) : (u,\theta) \text{ satisfies (7) – (9)}\}$, and $\mathcal{S} := \{(u,f) : (u,f) \text{ satisfies (11) – (14)}\}$. The following result shows that P0 and P1 are equivalent.

Lemma 1. Let $\mathcal{U}_0$ denote the set of all u's for which there exists θ such that $(u,\theta) \in \mathcal{Q}$, and let $\mathcal{U}_1$ denote the set of all u's for which there exists an f such that $(u,f) \in \mathcal{S}$. Then, $\mathcal{U}_0 \equiv \mathcal{U}_1$.

Proof. First, we show that $\mathcal{U}_0 \subseteq \mathcal{U}_1$. Suppose, $(\tilde{u},\tilde{\theta}) \in \mathcal{Q}$. Choose

$$\tilde{f}_{lj} = \gamma_{lj} g(\tilde{\theta}_l - \tilde{\theta}_j),$$

then, $\tilde{f} = \Gamma g(M^\top\tilde{\theta})$. By using (7), we have that $M\tilde{f} = M\Gamma g(M^\top\tilde{\theta}) = u$, i.e., $\tilde{f}$ satisfies (11). Because $\overline{f}_{lj} = -\underline{f}_{lj} = \gamma_{lj} g(\phi)$ and $\tilde{\theta}$ satisfies (9), $\tilde{f}$ satisfies (13). Also, along each $\mathcal{C}_i$, we have that

$$\sum_{(l,j)\in\vec{\mathcal{E}}_i} h\left(\tilde{f}_{lj}/\gamma_{lj}\right) = \sum_{(l,j)\in\vec{\mathcal{E}}_i} (\tilde{\theta}_l - \tilde{\theta}_j) = 0; \tag{18}$$

therefore, $(\tilde{u},\tilde{f}) \in \mathcal{S}$.

Now, we show that $\mathcal{U}_1 \subseteq \mathcal{U}_0$. Suppose, $(\tilde{u},\tilde{f}) \in \mathcal{S}$. We show how to construct $\tilde{\theta}$ from $\tilde{f}$. We assign any value to $\tilde{\theta}_1$, and compute, for each neighbor of node 1, $\tilde{\theta}_j = \tilde{\theta}_1 - h(\tilde{f}_{1j}/\gamma_{1j})$, $\forall j \in \mathcal{N}(1)$. Then, we perform the same computations for the neighbors of each $j \in \mathcal{N}(1)$, and continue this process. For cyclic networks, this computation continues until $\tilde{\theta}_l$ is computed from two different nodes i and k simultaneously. In order to successfully construct $\tilde{\theta}$, it is enough to show that $\tilde{\theta}_l$ as computed by node i, which is equal to $\tilde{\theta}_i - h(\tilde{f}_{il}/\gamma_{il})$, matches $\tilde{\theta}_l$ as computed by node k, which is equal to $\tilde{\theta}_k - h(\tilde{f}_{kl}/\gamma_{kl})$. Indeed, let $\mathcal{C}_1$ denote the cycle containing l, i and k, and suppose that traversing the path in the clockwise direction from i to k along $\vec{\mathcal{C}}_1$ does not go through node l, then, we have that

$$\tilde{\theta}_i - \tilde{\theta}_k = \sum_{(l,j)\in\vec{\mathcal{E}}_1'} h(\tilde{f}_{lj}/\gamma_{lj}), \tag{19}$$

where $\vec{\mathcal{E}}_1' = \vec{\mathcal{E}}_1 \backslash \{(k,l) \cup (l,i)\}$, and, therefore,

$$\tilde{\theta}_i - h\left(\frac{\tilde{f}_{il}}{\gamma_{il}}\right) - \tilde{\theta}_k + h\left(\frac{\tilde{f}_{kl}}{\gamma_{kl}}\right) = \sum_{(l,j)\in\vec{\mathcal{E}}_1'} h\left(\frac{\tilde{f}_{lj}}{\gamma_{lj}}\right) - h\left(\frac{\tilde{f}_{il}}{\gamma_{il}}\right) + h\left(\frac{\tilde{f}_{kl}}{\gamma_{kl}}\right) = 0, \tag{20}$$

where in the last equality we used the fact that $\tilde{f}$ satisfies the cycle constraint in (14). Therefore, $\tilde{\theta}_l$ as computed by node i matches $\tilde{\theta}_l$ as computed by node k. Then, obviously, the constraints in (7) – (9) are satisfied; therefore, $\mathcal{U}_0 \equiv \mathcal{U}_1$. □

D. Application Examples

The flow problem P0 finds different applications in electric power networks, water supply systems, and natural gas networks. In electric power networks, a simplified economic dispatch problem in which we only consider generation capacity constraints and line capacity constraints on active power flows, and ignore line losses, voltage constraints and reactive power flows, can be formulated as P0. Then, u_i corresponds to the active power injection at node $i \in \mathcal{V}$, and assuming losses can be neglected, $f_{lj} = \gamma_{lj} g(\theta_l - \theta_j)$ is the flow of active power along the electrical line $\{l, j\} \in \mathcal{E}$, where $g(\theta_l - \theta_j) = \sin(\theta_l - \theta_j)$, with θ_i being the phase angle of the voltage phasor at node i, and $\gamma_{lj} = V_l V_j B_{lj}$, with V_i being the voltage amplitude, and $-B_{lj}$ being the susceptance of the electrical line $\{l, j\}$.

In water supply networks, following the notation in [9], $-u_i$ corresponds to the water demand at node $i \in \mathcal{V}$, f_{lj} corresponds to the flow rate in pipe $\{l, j\}$, θ_i denotes the elevation head plus the pressure head at node i. Either the Darcy-Weisbach or the Hazen-Williams head-loss model is typically used to express $g(\cdot)$ and γ_{lj}, which are given by

$$g(\theta_l - \theta_j) = \mathrm{sgn}(\theta_l - \theta_j)|\theta_l - \theta_j|^{\frac{1}{\beta}},$$

and $\gamma_{lj} = \frac{1}{K_{lj}^{1/\beta}}$, where

$$K_{lj} = \frac{8 a_{lj} L_{lj}}{g \pi^2 D_{lj}^5}, \tag{21}$$

with $\beta = 2$, by the Darcy-Weisbach head-loss model, and

$$K_{lj} = \frac{b L_{lj}}{c^\beta D_{lj}^{4.87}}, \tag{22}$$

with $\beta = 1.85$, by the Hazen-Williams head-loss model, in which a_{lj} is the Darcy-Weisbach friction factor, b and c are the Hazen-Williams coefficients, L_{lj} and D_{lj} are the length and diameter of pipe $\{l, j\}$ (see, e.g., [12], [25]).

In natural gas networks, the economic dispatch problem can be formulated as P0 with

$$g(\theta_l - \theta_j) = \mathrm{sgn}(\theta_l - \theta_j)|\theta_l - \theta_j|^{0.5},$$

where the square root of θ_j corresponds to the gas pressure at node j, and γ_{lj} is some constant that depends on the pipeline parameters or gas properties [13].

III. Entropy-Like Penalty Functions for Solving the NFP Under Cycle Constraints

In this section, we formulate a convex approximation of the original problem P1, in which we add a penalty function to the cost function to penalize the violation of the cycle constraints. If the penalty function is minimized, the solution satisfies the cycle constraints provided the flows on the lines along cyclic paths are not binding.

First, we note that the set $\bigcup_{i=1}^{c} \mathcal{E}_i$ includes every edge that belongs to a cycle. Let $\mathcal{Z}$ denote the oriented counterpart of the set $\bigcup_{i=1}^{c} \mathcal{E}_i$, with the orientation consistent with that of the incidence matrix M, i.e. $(i, j) \in \mathcal{Z}$ only if $(i, j) \in \vec{\mathcal{E}}$. Then, we introduce the following *entropy-like function,*[1] denoted by $H(f)$, for the cycles in the network:

$$H(f) := -\sum_{(i,j)\in\mathcal{Z}} \int_0^{f_{ij}} \tilde{h}\left(\frac{x}{\gamma_{ij}}\right) dx, \tag{23}$$

where

$$\tilde{h}(y) = \begin{cases} -\phi - \kappa\epsilon^2, & y \leq \underline{y}, \\ h(y) - \kappa(y + \underline{y}_\epsilon)^2, & \underline{y} \leq y \leq \underline{y}_\epsilon, \\ h(y), & \underline{y}_\epsilon \leq y \leq \overline{y}_\epsilon, \\ h(y) + \kappa(y - \overline{y}_\epsilon)^2, & \overline{y}_\epsilon \leq y \leq \overline{y}, \\ \phi + \kappa\epsilon^2, & \overline{y} \leq y, \end{cases} \tag{24}$$

with $\overline{y} := g(\phi)$, $\underline{y} := -g(\phi)$, $\overline{y}_\epsilon := g(\phi) - \epsilon$, $\underline{y}_\epsilon := -g(\phi) + \epsilon$, for some parameters $\epsilon > 0$, and $\kappa > 0$ (to be discussed later). From (24), it can be seen that $H(f)$ penalizes the flows violating the box constraints for lines along cycles, where κ is the parameter for increasing the slope of the penalty. An example of $\tilde{h}(y)$ and the entropy-like component for line (i, j), denoted by

$$H_{ij}(f_{ij}) := -\int_0^{f_{ij}} \tilde{h}\left(\frac{x}{\gamma_{ij}}\right) dx,$$

is given in Figure 1. In the remainder, we relax the non-convex cycle constraints and box constraints on the flows along the cycles, and instead maximize the entropy-like function $H(f)$. To be more precise, to deal with the non-convexity in P1, we add $H(f)$ as a penalty term to the cost function to replace the cycle and box constraints in (13) – (14) for flows along the cycles; this results in the following optimization problem:

$$\mathbf{P2}: \min_{u\in\mathbb{R}^n, f\in\mathbb{R}^\ell} \sum_{i\in\mathcal{V}} F_i(u_i) - \rho H(f) \tag{25}$$

$$\text{subject to } u = Mf, \tag{26}$$

$$\underline{u} \leq u \leq \overline{u}, \tag{27}$$

$$\underline{f}_{ij} \leq f_{ij} \leq \overline{f}_{ij}, \ (i,j) \in \vec{\mathcal{E}} \setminus \mathcal{Z}, \tag{28}$$

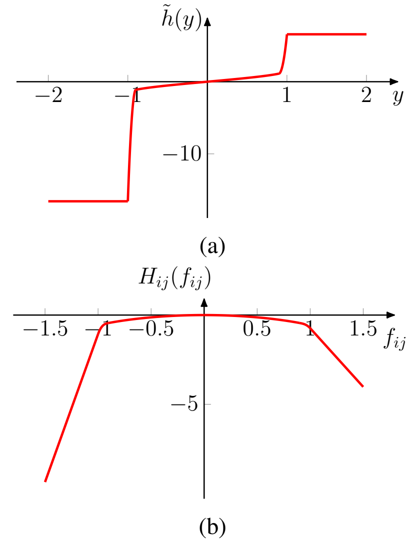

Fig. 1: (a) Plot of $\tilde{h}(y)$ for $\gamma_{ij} = 1$, $\epsilon = 0.1$, $\kappa = 5 \times 10^3$, and (b) corresponding entropy-like component $H_{ij}(f_{ij})$.

[1] The function $H(f)$ has certain properties typical of entropy functions; for example, the negative of $H(f)$ is a convex integral, $H(f)$ is concave, and maximizing $H(f)$ allows to achieve a system-wide objective, which is, in our case, a satisfaction of the non-convex cycle constraints in (14) (to be shown later) [20], [26], [27]. This is the reason why we use the term entropy-like function to refer to $H(\cdot)$.

where $\rho > 0$ is the penalty weight. In (28), we only have the constraints on the flows along edges that do not belong to any cycle. The result in the next lemma establishes that P2 is convex.

Lemma 2. $H(f)$ defined in (23) is concave, and P2 is convex.

Now, we state one of the main results of this work.

Proposition 1. Let $(u^{(1)*}, f^{(1)*})$ and $(u^{(2)*}, f^{(2)*})$ denote the solutions of P1 and P2, respectively. If

$$\kappa = \frac{3}{\rho\epsilon^3}\Big(\sum_{i\in\mathcal{V}} F_i(\overline{u}_i) - \rho H(\overline{f} - \epsilon\gamma)\Big), \tag{29}$$

then, the minimum of P2, $(u^{(2)*}, f^{(2)*})$, satisfies the cycle constraints in (14) if $\underline{f}_{ij} + \epsilon\gamma_{ij} \leq f^{(2)*}_{ij} \leq \overline{f}_{ij} - \epsilon\gamma_{ij}, \forall (i,j) \in \mathcal{Z}$, and the box constraints for flows in (13) if $\underline{f}_{ij} + \epsilon\gamma_{ij} \leq f^{(1)*}_{ij} \leq \overline{f}_{ij} - \epsilon\gamma_{ij}, \forall(i,j) \in \mathcal{Z}$.

Proof. Consider the Lagrangian for problem P2:

$$\begin{aligned} L(u, f, \lambda, \mu, \nu, a, b) = &\sum_{i\in\mathcal{V}} F_i(u_i) - \rho H(f) + \lambda^\top(u - Mf) \\ &+ \mu^\top(\underline{u} - u) + \nu^\top(u - \overline{u}) \\ &+ \sum_{(i,j)\in\vec{\mathcal{E}}\setminus\mathcal{Z}} \Big(a_{ij}(\underline{f}_{ij} - f_{ij}) \\ &+ b_{ij}(f_{ij} - \overline{f}_{ij})\Big), \end{aligned} \tag{30}$$

where λ, μ, ν, a_{ij}'s and b_{ij}'s are the dual variables, $a = [\{a_{ij}\}_{(i,j)\in\vec{\mathcal{E}}\setminus\mathcal{Z}}]^\top$, $b = [\{b_{ij}\}_{(i,j)\in\vec{\mathcal{E}}\setminus\mathcal{Z}}]^\top$. By denoting λ^*, μ^*, ν^*, a^*_{ij}'s and b^*_{ij}'s as optimal dual variables, we can write the KKT conditions as follows:

$$\frac{\partial}{\partial u}\Big(L(u^{(2)*}, f^{(2)*}, \lambda^*, \mu^*, \nu^*, a^*, b^*)\Big) = 0, \tag{31}$$

$$\frac{\partial}{\partial f}\Big(L(u^{(2)*}, f^{(2)*}, \lambda^*, \mu^*, \nu^*, a^*, b^*)\Big) = 0, \tag{32}$$

$$(\underline{u} - u^{(2)*})^\top\mu^* = 0, \tag{33}$$

$$(u^{(2)*} - \overline{u})^\top\nu^* = 0, \tag{34}$$

$$a^*_{ij}(\underline{f}_{ij} - f^{(2)*}_{ij}) = 0, (i,j) \in \vec{\mathcal{E}} \setminus \mathcal{Z}, \tag{35}$$

$$b^*_{ij}(f^{(2)*}_{ij} - \overline{f}_{ij}) = 0, \tag{36}$$

where $a^* = [\{a^*_{ij}\}_{(i,j)\in\vec{\mathcal{E}}\setminus\mathcal{Z}}]^\top$, $b^* = [\{b^*_{ij}\}_{(i,j)\in\vec{\mathcal{E}}\setminus\mathcal{Z}}]^\top$. From (32), we have that

$$z - M^\top\lambda = 0, \tag{37}$$

where

$$z_e = \begin{cases} \rho\tilde{h}\Big(\frac{f^{(2)*}_{ij}}{\gamma_{ij}}\Big) & \text{if } (i,j) \in \mathcal{Z}, \\ -a^*_{ij} + b^*_{ij} & \text{if } (i,j) \in \vec{\mathcal{E}} \setminus \mathcal{Z}, \end{cases} \tag{38}$$

with $e = \mathbb{I}(\{i,j\})$. By using (2), it follows that

$$0 = (n^{(i)})^\top(z - M^\top\lambda) = (n^{(i)})^\top z, \tag{39}$$

for any cycle $\mathcal{C}_i$, and by using (38), we obtain

$$\sum_{(l,j)\in\vec{\mathcal{E}}_i} \tilde{h}\left(f^{(2)*}_{lj}/\gamma_{lj}\right) = 0. \tag{40}$$

Note from the definition of $\tilde{h}(\cdot)$ in (24) that if $\underline{f}_{lj} + \epsilon\gamma_{lj} \leq f^{(2)*}_{lj} \leq \overline{f}_{lj} - \epsilon\gamma_{lj}$, $\forall (l,j) \in \mathcal{Z}$, then,

$$\tilde{h}\left(f^{(2)*}_{lj}/\gamma_{lj}\right) = h\left(f^{(2)*}_{lj}/\gamma_{lj}\right),$$

and, therefore,

$$\sum_{(l,j)\in\vec{\mathcal{E}}_i} h\left(f^{(2)*}_{lj}/\gamma_{lj}\right) = 0, \tag{41}$$

hence, $f^{(2)*}$ satisfies the cycle constraints in (14).

Now, we show by contradiction that $f^{(2)*}$ satisfies the box constraints in (13). Suppose, on the contrary, that there exists a set of flows on cycle $\mathcal{C}_i$, denoted by

$$\mathcal{S}_i := \{(l,j) \in \vec{\mathcal{E}}_i : |f^{(2)*}_{lj}| > \gamma_{lj}g(\phi)\},$$

which violate the box constraints in (13). Then, from the definitions of $H(\cdot)$ and $\tilde{h}(\cdot)$ in (23) and (24) it follows that

$$\begin{aligned} H(f^{(2)*}) &= -\sum_{(l,j)\in\mathcal{S}_i} \int_0^{f^{(2)*}_{lj}} \tilde{h}\left(\frac{x}{\gamma_{lj}}\right)dx - \sum_{(l,j)\in\mathcal{Z}\setminus\mathcal{S}_i} \int_0^{f^{(2)*}_{lj}} \tilde{h}\left(\frac{x}{\gamma_{lj}}\right)dx \\ &\leq -\sum_{(l,j)\in\mathcal{S}_i} \int_0^{f^{(2)*}_{lj}} \tilde{h}\left(\frac{x}{\gamma_{lj}}\right)dx \\ &\leq -\sum_{(l,j)\in\mathcal{S}_i} \int_0^{\overline{y}} h(y)dy - \sum_{(l,j)\in\mathcal{S}_i} \int_{\overline{y}_\epsilon}^{\overline{y}} \kappa(y - \overline{y}_\epsilon)^2 dy \\ &< -\sum_{(l,j)\in\mathcal{S}_i} \int_{\overline{y}_\epsilon}^{\overline{y}} \kappa(y - \overline{y}_\epsilon)^2 dy \leq -\frac{\kappa}{3}\epsilon^3, \end{aligned} \tag{42}$$

where we use the fact that $\int_a^b h(y)dy > 0$, $b > a$, $\forall a, b \in \mathbb{R}$.

By using (42), we obtain the lower bound on the optimal cost of P2 as follows:

$$\sum_{i\in\mathcal{V}} F_i(u^{(2)*}_i) - \rho H(f^{(2)*}) \geq \sum_{i\in\mathcal{V}} F_i(u^{(2)*}_i) + \rho\frac{\kappa}{3}\epsilon^3 \tag{43}$$

$$> \sum_{i\in\mathcal{V}} F_i(\overline{u}_i) - \rho H(\overline{f} - \epsilon\gamma) \geq \sum_{i\in\mathcal{V}} F_i(u^{(1)*}_i) - \rho H(\overline{f} - \epsilon\gamma) \tag{44}$$

$$\geq \sum_{i\in\mathcal{V}} F_i(u^{(1)*}_i) - \rho H(f^{(1)*}), \tag{45}$$

where, to arrive at the inequality in (43), we used (42); by using (29) the inequality in (44) follows because $F_i(\cdot)$ is strictly increasing, and the last inequality follows from the fact that $H(\overline{f} - \epsilon\gamma) \leq H(f^{(1)*})$ because $\underline{f}_{ij} + \epsilon\gamma_{ij} \leq f^{(1)*}_{ij} \leq \overline{f}_{ij} - \epsilon\gamma_{ij}$, $\forall (i,j) \in \mathcal{Z}$. The lower bound given in (45) contradicts the fact that $(u^{(2)*}, f^{(2)*})$ is the minimum of P2. Thus, we have shown that $f^{(2)*}$ satisfies the box constraints in (13). □

We note that finding $(u^{(1)*}, f^{(1)*})$ is an NP-hard problem. Proposition 1 establishes that the solution of P2, $(u^{(2)*}, f^{(2)*})$, is a feasible point of the non-convex constraint set of P1, if $f^{(1)*}$ and $f^{(2)*}$ strictly satisfy the box constraints along each cycle. But satisfaction of this condition is not guaranteed by the proposed approach. Moreover, when flows on many lines are binding, i.e., the box constraints are not strictly satisfied, the proposed approach is not guaranteed to converge to the solution that satisfy the cycle constraints of P1.

We also note that $u^{(2)*}$ can be made as close to $u^{(1)*}$ as possible by making the penalty weight ρ sufficiently small, as established in the next lemma. However, having too small ρ slows down the convergence to the solution that satisfies the cycle constraints.

Lemma 3. Let $(u^{(1)*}, f^{(1)*})$ and $(u^{(2)*}, f^{(2)*})$ denote the solutions of P1 and P2, respectively. If $\underline{f}+\epsilon\gamma \le f^{(1)*} \le \overline{f}-\epsilon\gamma$, then,

$$\sum_{i\in\mathcal{V}} F_i(u_i^{(2)*}) - \sum_{i\in\mathcal{V}} F_i(u_i^{(1)*}) \le \rho(H(f^{(2)*}) - H(\overline{f}-\epsilon\gamma)).$$

IV. Entropy Maximization for Finding Feasible Flows for Cyclic Networks

Here, we consider a modified version of P1 obtained by dropping the cycle constraints in (14). Then, we propose a method to recover the solution of P1 from the solution of its modified version. The method is based on finding the flows that maximize the individual entropies of the cycles.

A. Finding Feasible Flows

Let $(u^{(3)*}, f^{(3)*})$ denote an optimal solution to a modified version of P1, referred to as P3, where the cycle constraints in (14) have been removed:

$$\textbf{P3}: \min_{u\in\mathbb{R}^n, f\in\mathbb{R}^\ell} \sum_{i\in\mathcal{V}} F_i(u_i) \tag{46}$$

$$\text{subject to } u = Mf, \tag{47}$$

$$\underline{u} \le u \le \overline{u}, \tag{48}$$

$$\underline{f} \le f \le \overline{f}. \tag{49}$$

Let $(u^{(3)*}, f^{*})$ denote another minimizer of P3 such that f^* also satisfies the cycle constraints in (14). Then, it is easy to see from (4) that $f^* = f^{(3)*} + N^\top\mu^*$ for some $\mu^* \in \mathbb{R}^c$, where μ_i^* corresponds to cycle $\mathcal{C}_i$. We also note that since $(u^{(3)*}, f^{(3)*})$ is a minimum of P3, $(u^{(3)*}, f^{*})$ is a minimum of P1. In the following, we show how to construct f^* from $f^{(3)*}$ by computing μ^*. The problem of obtaining μ^* will be set up as an entropy maximization problem.

Consider a connected subgraph of $\mathcal{G}$ denoted by $\mathcal{G}_i$, in which each edge belongs to a cycle. $\mathcal{G}_i$ is referred to as being *completely cyclic*. Suppose $\mathcal{G}_i$ is also *maximal completely cyclic*, i.e., it is not a proper subgraph of some other connected completely cyclic subgraph. Since $\mathcal{G}$ might contain several maximal completely cyclic subgraphs that are edge disjoint, we denote them as $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_w$, where w is the total number of maximal completely cyclic subgraphs in $\mathcal{G}$.

Define $\mathcal{A}_i = \{j : \mathcal{C}_j \subseteq \mathcal{G}_i\}$. We introduce the following entropy-like function, denoted by $H_i(\cdot)$, for the maximal completely cyclic $\mathcal{G}_i$:

$$H_i(\mu^{(i)}) = -\sum_{e=1}^{\ell} \int_0^{\sum_{a\in\mathcal{A}_i} n_e^{(a)}\mu_a} \tilde{h}\left(\frac{f_e^{(3)*}+x}{\gamma_e}\right) dx, \tag{50}$$

where $\mu^{(i)} = [\{\mu_j\}_{j\in\mathcal{A}_i}]^\top$, and $\tilde{h}(\cdot)$ is redefined as follows:

$$\tilde{h}(y) = \begin{cases} h(y), & \text{if } y \in \mathcal{Y}, \\ \phi + y - g(\phi), & y \ge g(\phi), \\ -\phi - y + g(\phi), & y \le -g(\phi). \end{cases} \tag{51}$$

Since $\tilde{h}(\cdot)$ is continuous everywhere, for $k \in \mathcal{A}_i$, we have that

$$\frac{\partial H_i(\mu^{(i)})}{\partial \mu_k} = -\sum_{e=1}^{\ell} n_e^{(k)} \tilde{h}\left(\frac{f_e^{(3)*} + \sum_{a\in\mathcal{A}_i} n_e^{(a)}\mu_a}{\gamma_e}\right) \tag{52}$$

is continuous everywhere, and therefore, $H_i(\mu^{(i)})$ is continuously differentiable everywhere. In the next lemma, we establish that the entropy-like function $H_i(\mu^{(i)})$ is strictly concave.

Lemma 4. Consider the entropy-like function $H_i(\mu^{(i)})$ defined in (50) for the maximal completely cyclic subgraph $\mathcal{G}_i$. Then, $H_i(\mu^{(i)})$ is strictly concave.

Proof. By using (50) and the fact that when $\tilde{h}(x)$ is monotonically increasing, $\int_a^b \tilde{h}(x)dx > \tilde{h}(a)(b-a)$, $\forall a, b \in \mathbb{R}$, we obtain that

$$\begin{aligned} H_i(\mu^{(i)} + \Delta\mu^{(i)}) &= H_i(\mu^{(i)}) - \sum_{e=1}^{\ell} \int_{x_{1e}}^{x_{2e}} \tilde{h}\left(\frac{f_e^{(3)*}+x}{\gamma_e}\right) dx \\ &< H_i(\mu^{(i)}) - \sum_{e=1}^{\ell} \sum_{k\in\mathcal{A}_i} n_e^{(k)} \Delta\mu_k \tilde{h}\left(\frac{f_e^{(3)*} + \sum_{a\in\mathcal{A}_i} n_e^{(a)}\mu_a}{\gamma_e}\right) \\ &= H_i(\mu^{(i)}) - \sum_{k\in\mathcal{A}_i} \Delta\mu_k \sum_{e=1}^{\ell} n_e^{(k)} \tilde{h}\left(\frac{f_e^{(3)*} + \sum_{a\in\mathcal{A}_i} n_e^{(a)}\mu_a}{\gamma_e}\right) \\ &= H_i(\mu^{(i)}) + \nabla H_i(\mu^{(i)})^\top \Delta\mu^{(i)}, \end{aligned} \tag{53}$$

where $x_{1e} \coloneqq \sum_{a\in\mathcal{A}_i} n_e^{(a)}\mu_a$, $x_{2e} \coloneqq \sum_{a\in\mathcal{A}_i} n_e^{(a)}(\mu_a + \Delta\mu_a)$, and in the last equality we used (52). Thus, the first-order concavity condition is strictly satisfied, and $H_i(\mu^{(i)})$ is strictly concave. ☐

For a given $(u^{(3)*}, f^{(3)*})$ that solves P3, the next proposition shows that in order to obtain μ^* so that $(u^{(3)*}, f^{(3)*} + N^\top\mu^*)$ solves P1, we need to maximize the entropy for each maximal completely cyclic subgraph $\mathcal{G}_i$.

Proposition 2. Let $\mu^{(i*)} = [\{\mu_k^*\}_{k\in\mathcal{A}_i}]^\top$, and

$$\mathcal{D}_k = \{x \in \mathbb{R} : \underline{f}_{lj} \le f_{lj}^{(3)*} + x \le \overline{f}_{lj}, \forall (l,j) \in \vec{\mathcal{E}}_k\}.$$

Then, $(u^{(3)*}, f^{(3)*} + N^\top\mu^*)$ is a solution of P1 if, for any $\mathcal{G}_i$, $i = 1, 2\ldots, w$,

$$\mu^{(i*)} = \operatorname{argmax}_{\mu^{(i)}\in\mathbb{R}^{|\mathcal{A}_i|}} H_i(\mu^{(i)}), \tag{54}$$

and $\mu_k^* \in \mathcal{D}_k$, $\forall k \in \mathcal{A}_i$.

Proof. From (52), note that

$$\frac{\partial H_i(\mu^{(i)})}{\partial \mu_k} = -\sum_{e=1}^{\ell} n_e^{(k)} h\left(\frac{f_e^{(3)*} + \sum_{j\in\mathcal{A}_i} n_e^{(j)}\mu_j}{\gamma_e}\right) \quad (55)$$

if $\mu_k \in \mathcal{D}_k$, $\forall k \in \mathcal{A}_i$; then, in this case, $\nabla H_i(\mu^{(i)}) = 0$ are the cycle constraints for cycles in $\mathcal{G}_i$, which are satisfied at $\mu_k = \mu_k^*$, $\forall k \in \mathcal{A}_i$, i.e.,

$$\nabla H_i(\mu^{(i*)}) = 0,\ \forall \mathcal{G}_i,\ i = 1, 2 \ldots, w, \quad (56)$$

if and only if $(u^{(3)*}, f^{(3)*} + N^\top \mu^*)$ is a solution of P1. Since, by Lemma 4, $H_i(\mu^{(i)})$ is concave, the first-order concavity conditions in (56) are satisfied if $H_i(\mu^{(i)})$ achieves a maximum at $\mu^{(i)} = \mu^{(i*)}$, $\forall \mathcal{G}_i$. □

V. Convex Approximation of the Non-Convex Cycle Constraints

In the previous section, we showed how to construct f^*, which satisfies the cycle constraints in (14), from the solution of P3. In this section, we give sufficient conditions (which might also be necessary in some cases) for the existence of f^* for certain types of cyclic networks. These conditions allow to obtain a convex inner approximation of the original non-convex constraint set in P1, and to recast the original non-convex problem as a convex problem. The main idea behind this inner approximation is roughly based on lowering the capacities of the lines along cyclic paths.

A. Networks with Edge-Disjoint Cycles

First, we consider a network with edge-disjoint cycles, i.e., any pair of cycles have no edges in common. For each cycle $\mathcal{C}_i$, we define the following functions:

$$h_i(f) := \sum_{(l,j)\in\vec{\mathcal{E}}_i} h\left(\frac{f_{lj}}{\gamma_{lj}}\right), \quad (57)$$

$$\underline{\mu}_i(f) := \max_{(l,j)\in\vec{\mathcal{E}}_i} (\underline{f}_{lj} - f_{lj}), \quad (58)$$

$$\overline{\mu}_i(f) := \min_{(l,j)\in\vec{\mathcal{E}}_i} (\overline{f}_{lj} - f_{lj}). \quad (59)$$

Recall that $\mathcal{S}$ denotes the constraint set of P1, and $(u^{(3)*}, f^{(3)*})$ denotes the solution of P3. We state the following result, which extends the single cycle feasibility lemma in [28, page 6, Lemma 4] to the case when there are multiple edge-disjoint cycles.

Lemma 5. Consider a network with edge-disjoint cycles, $\mathcal{C}_1, \ldots, \mathcal{C}_c$. If $f^{(3)*}$ satisfies the following two inequalities for each cycle $\mathcal{C}_i$:

$$h_i(f^{(3)*} + n^{(i)}\overline{\mu}_i(f^{(3)*})) \geq 0, \quad (60)$$

and

$$h_i(f^{(3)*} + n^{(i)}\underline{\mu}_i(f^{(3)*})) \leq 0, \quad (61)$$

then, there exists f^* such that $(u^{(3)*}, f^*) \in \mathcal{S}$.

Proof. Suppose $\mu^{(1)}$ and $\mu^{(2)}$ are some constants and $\mu^{(1)} < \mu^{(2)}$. Since $h(\cdot)$ is a monotonically increasing function, for any cycle $\mathcal{C}_i$, we have that

$$\begin{aligned} h_i(f^{(3)*} + n^{(i)}\mu^{(1)}) &= \sum_{(l,j)\in\vec{\mathcal{E}}_i} h\left(\frac{f_{lj}^{(3)*} + \mu^{(1)}}{\gamma_{lj}}\right) \\ &< \sum_{(l,j)\in\vec{\mathcal{E}}_i} h\left(\frac{f_{lj}^{(3)*} + \mu^{(2)}}{\gamma_{lj}}\right) \\ &= h_i(f^{(3)*} + n^{(i)}\mu^{(2)}), \end{aligned} \quad (62)$$

therefore, $h_i(f^{(3)*} + n^{(i)}\mu)$ is monotonically increasing in μ. Since

$$h_i(f^{(3)*} + n^{(i)}\underline{\mu}_i(f^{(3)*})) \leq 0 \leq h_i(f^{(3)*} + n^{(i)}\overline{\mu}_i(f^{(3)*})),$$

there exists $\mu_i \in [\underline{\mu}_i, \overline{\mu}_i]$ such that

$$h_i(f^{(3)*} + n^{(i)}\mu_i) = 0, \quad i = 1, \ldots, c.$$

Hence, it follows that $(u^{(3)*}, f^{(3)*} + N^\top \mu) \in \mathcal{S}$. □

Conditions in (60) – (61) are necessary and sufficient for f^* to exist, and they can be checked after solving P3 to decide whether or not the optimal injections, u^*, are a feasible solution for P1. Nevertheless, what is more desirable here is to be able to enforce conditions in (60) – (61) while solving P3. In this regard, we introduce a set of constraints that will be added to P3 to ensure that the new solution belongs to $\mathcal{S}$.

For $i = 1, \ldots, c$, and $\beta \geq 0$, define

$$\begin{aligned} \mathcal{F}_i(\beta) = \{f : h_i(f + n^{(i)}\overline{\mu}_i(f)) \geq 0,\ h_i(f + n^{(i)}\underline{\mu}_i(f)) \leq 0, \\ \overline{\mu}_i(f) \geq \beta,\ \underline{\mu}_i(f) \leq -\beta\}. \end{aligned} \quad (63)$$

We note that although $\mathcal{F}_i(0)$ is non-convex, with appropriate β, $\mathcal{F}_i(\beta)$ becomes convex, as established in the next proposition.

Proposition 3. Let d_i denote the number of edges in cycle $\mathcal{C}_i$, and define

$$\overline{\gamma}_i := \max_{(l,j)\in\vec{\mathcal{E}}_i} \overline{f}_{lj}, \underline{\gamma}_i := \min_{(l,j)\in\vec{\mathcal{E}}_i} \overline{f}_{lj}, \text{ and } \psi_i := \frac{\phi}{d_i - 1}.$$

If

$$\beta_i^* = \frac{1}{2}\overline{\gamma}_i - \frac{\underline{\gamma}_i}{2} g(\psi_i), \quad (64)$$

then,

$$\mathcal{F}_i(\beta_i^*) \equiv \mathcal{B}_i := \{f : \underline{f}_{lj} + \beta_i^* \leq f_{lj} \leq \overline{f}_{lj} - \beta_i^*, \forall (l,j) \in \vec{\mathcal{E}}_i\}.$$

Proof. Define

$$\mathcal{M}_i = \{f : \overline{\mu}_i(f) \geq \beta_i^*, \underline{\mu}_i(f) \leq -\beta_i^*\};$$

next, we show that $\mathcal{B}_i \equiv \mathcal{M}_i$. Suppose that $f \in \mathcal{B}_i$. Since

$$f_{lj} \leq \overline{f}_{lj} - \beta_i^*, \quad \forall (l,j) \in \vec{\mathcal{E}}_i,$$

by definition in (59), it follows that $\overline{\mu}_i(f) \geq \beta_i^*$. Similarly, $\underline{\mu}_i(f) \leq -\beta_i^*$. Conversely, if $f \in \mathcal{M}_i$, then,

$$\overline{f}_{lj} - f_{lj} \geq \overline{\mu}_i(f) \geq \beta_i^*,$$

and, therefore, $f_{lj} \leq \overline{f}_{lj} - \beta_i^*$. Similarly, if $\underline{\mu}_i(f) \leq -\beta_i^*$, then, $\underline{f}_{lj} + \beta_i^* \leq f_{lj}$; therefore, $\mathcal{B}_i \equiv \mathcal{M}_i$.

Now, we show that $\mathcal{F}_i(\beta_i^*) \equiv \mathcal{B}_i$. Because $\mathcal{F}_i(\beta_i^*) \subseteq \mathcal{M}_i$, it is enough to prove that, $\forall f \in \mathcal{M}_i$,

$$h_i(f + n^{(i)}\overline{\mu}_i(f)) \geq 0 \text{ and } h_i(f + n^{(i)}\underline{\mu}_i(f)) \leq 0.$$

Suppose $f \in \mathcal{M}_i$, then, by using (64) and the fact that $\mathcal{B}_i \equiv \mathcal{M}_i$, we have that, for $(l,j) \in \vec{\mathcal{E}}_i$,

$$f_{lj} + \overline{\mu}_i(f) \geq \underline{f}_{lj} + 2\beta_i^* = \underline{f}_{lj} + \overline{\gamma}_i - \underline{\gamma}_i g(\psi_i) \geq -\gamma_{lj} g(\psi_i). \tag{65}$$

Define $(a,b) = \mathrm{argmin}_{(l,j)\in\vec{\mathcal{E}}_i}(\overline{f}_{lj} - f_{lj})$ and $\mathcal{J}_i = \vec{\mathcal{E}}_i \setminus (a,b)$. Then,

$$h_i(f + n^{(i)}\overline{\mu}_i(f)) = \phi + \sum_{(l,j)\in\mathcal{J}_i} h\left(f_{lj} + \overline{\mu}_i(f)\right).$$

By using (65) and the fact that $h(\cdot)$ is monotonically increasing, we obtain that

$$h_i(f + n^{(i)}\overline{\mu}_i(f)) \geq \phi - \sum_{(l,j)\in\mathcal{J}_i} h(g(\psi_i)) = 0.$$

Similary, $h_i(f + n^{(i)}\underline{\mu}_i(f)) \leq 0$; therefore, $\mathcal{F}_i(\beta_i^*) \equiv \mathcal{B}_i$. □

In view of Proposition 3, an additional constraint, $f \in \mathcal{F}_i(\beta_i^*)$, can be added to P3 to approximate the non-convex cycle constraint in P1:

$$\textbf{P}4: \min_{u\in\mathbb{R}^n, f\in\mathbb{R}^\ell} \sum_{i\in\mathcal{V}} F_i(u_i) \tag{66}$$

$$\text{subject to } u = Mf, \tag{67}$$

$$\underline{u} \leq u \leq \overline{u}, \tag{68}$$

$$\underline{f} \leq f \leq \overline{f}, \tag{69}$$

$$f \in \mathcal{F}_i(\beta_i^*), \quad i = 1, \ldots, c, \tag{70}$$

where the cycle constraint in (70) is convex.

Remark 2. The problem of evaluating the accuracy of the convex inner approximation is a very hard problem in general. Later, through simulations, we show that this approximation is accurate enough for practical use.

The proposed convex approximation in (70) might become conservative in the following scenario. By Proposition 3, if d_i is very large, we have that $\psi_i \approx 0$, and $\beta_i^* = \frac{1}{2}\overline{\gamma}_i$. If we choose $(l,j) \in \vec{\mathcal{E}}_i$ such that $\overline{f}_{lj} = \underline{\gamma}_i$, then, the constraint in (70) for edge (l,j) reads as $-\underline{\gamma}_i + \frac{1}{2}\overline{\gamma}_i \leq f_{lj} \leq \underline{\gamma}_i - \frac{1}{2}\overline{\gamma}_i$, which is not feasible if $\underline{\gamma}_i \leq \frac{1}{2}\overline{\gamma}_i$. Thus, the result is more conservative when a cycle has many edges and the values of flow limits along the edges are spread over a large interval.

B. General Networks with Cycles

Here, we consider a more general case when cycles are not necessarily edge-disjoint. Let $\mathcal{O}$ denote the set of cycles, in which each cycle shares an edge with at least one cycle, and let $\mathcal{O}_i$ denote the set of cycles, which share an edge with cycle $\mathcal{C}_i$. To formulate a convex problem similar to P4, we need the result in the following proposition.

Proposition 4. Consider a network with fundamental cycles, $\mathcal{C}_1, \ldots, \mathcal{C}_c$, and $|\mathcal{O}_i| \leq 1$, $i = 1, \ldots, c$. Suppose, for any two cycles $\mathcal{C}_i$ and $\mathcal{C}_j$ with a common edge $\{x,y\}$, we have that

$$\underline{f}_{xy} + \beta_i^* + \beta_j^* \leq f_{xy}^{(3)*} \leq \overline{f}_{xy} - \beta_i^* - \beta_j^*, \tag{71}$$

where β_k^*, $k = 1, \ldots, c$, is the same as in (64). Then, there exists $\mu^* \in \mathbb{R}^c$ such that $(u^{(3)*}, f^{(3)*} + N^\top \mu^*) \in \mathcal{S}$.

Proof. We define $h_i(\cdot)$ differently for cycles that are not edge-disjoint as shown below:

$$h_i(f^{(3)*} + n^{(i)}\mu_i, \mu_j) = \sum_{(l,j)\in\vec{\mathcal{E}}'_i} h\left(\frac{f_{lj}^{(3)*} + \mu_i}{\gamma_{lj}}\right) + n_p^{(i)} h\left(\frac{f_p^{(3)*} + n_p^{(i)}\mu_i + n_p^{(j)}\mu_j}{\gamma_p}\right),$$

where $p = \mathbb{I}(\{x,y\})$, $\vec{\mathcal{E}}'_i = \vec{\mathcal{E}}_i \backslash (x,y)$ if $(x,y) \in \vec{\mathcal{E}}_i$, and $\vec{\mathcal{E}}'_i = \vec{\mathcal{E}}_i \backslash (y,x)$, otherwise. Similar to the proof of Lemma 5, for fixed μ_j, we have that $h_i(f^{(3)*} + n^{(i)}\mu_i, \mu_j)$ is monotonically increasing in μ_i. Suppose $z = r_i(\mu_j)$ is a root of $h_i(f^{(3)*} + n^{(i)}z, \mu_j) = 0$ as a function of μ_j; next, we show that $\mu_i^* \in [-\beta_i^*, \beta_i^*]$ and $\mu_j^* \in [-\beta_j^*, \beta_j^*]$. Consider any arbitrary μ_i and μ_j such that $\mu_i \in [-\beta_i^*, \beta_i^*]$ and $\mu_j \in [-\beta_j^*, \beta_j^*]$. Then, similar to the proof of Proposition 3, it is easy to show that

$$h_i(f^{(3)*} + n^{(i)}\beta_i^*, \mu_j) \geq 0, \text{ and } h_i(f^{(3)*} - n^{(i)}\beta_i^*, \mu_j) \leq 0.$$

Since $h_i(f^{(3)*} + n^{(i)}\mu_i, \mu_j)$ is monotonically increasing in μ_i, we have that

$$-\beta_i^* \leq r_i(\mu_j) \leq \beta_i^*.$$

Then, by a similar argument, we conclude that

$$-\beta_j^* \leq r_j(\mu_i) \leq \beta_j^*.$$

Therefore, $\mu_i^* \in [-\beta_i^*, \beta_i^*]$ and $\mu_j^* \in [-\beta_j^*, \beta_j^*]$. Also, it is easy to see that

$$\underline{f} \leq f^{(3)*} + n^{(i)}\mu_i^* + n^{(j)}\mu_j^* \leq \overline{f}.$$

Thus, $(u^{(3)*}, f^{(3)*} + N^\top \mu^*) \in \mathcal{S}$. □

The following proposition extends the results of Proposition 4 to the more general case when each edge belongs to at most two cycles; we omit the proof as it is similar to that of Proposition 4.

Proposition 5. Consider a network with fundamental cycles $\mathcal{C}_1, \ldots, \mathcal{C}_c$, where each edge belongs to at most two cycles. Suppose, for any two cycles $\mathcal{C}_i$ and $\mathcal{C}_j$ with a common edge $\{x,y\}$, we have that

$$\underline{f}_{xy} + \beta_i^* + \beta_j^* \leq f_{xy}^{(3)*} \leq \overline{f}_{xy} - \beta_i^* - \beta_j^*, \tag{72}$$

where β_k^*, $k = 1, \ldots, c$, is the same as in (64). Then, there exists $\mu^* \in \mathbb{R}^c$ such that $(u^{(3)*}, f^{(3)*} + N^\top \mu^*) \in \mathcal{S}$.

For the case when each edge belongs to at most two cycles, we formulate another convex problem, referred to as P5, by adding additional constraint in (75) into P4:

$$\textbf{P}5: \min_{u\in\mathbb{R}^n, f\in\mathbb{R}^\ell} \sum_{i\in\mathcal{V}} F_i(u_i) \tag{73}$$

$$\text{subject to (67) – (70)} \tag{74}$$

$$\underline{f}_{xy} + \beta_i^* + \beta_j^* \leq f_{xy} \leq \overline{f}_{xy} - \beta_i^* - \beta_j^*,$$
$$\forall\{x,y\} \in \mathcal{E}_i \cap \mathcal{E}_j, \forall i,j. \tag{75}$$

The general case with each edge belonging to an arbitrary

TABLE I: Values of the cost $F(u)$ and ϵ for the Hanoi water distribution network.

ρ	$F(u)$	ϵ (m)
Global solution	1.199×10^6	0
0	1.199×10^6	34.96
0.001	1.199×10^6	0.22
0.01	1.200×10^6	0.031
0.1	1.237×10^6	1.5×10^{-3}
10	1.809×10^6	9.6×10^{-5}

number of cycles can be treated in a similar way, which, however, yields inner approximation of the constraint set with a low accuracy. More accurate inner approximation can be constructed by using sharper bounds for β_i^*'s.

VI. Simulations

In this section, we perform numerical simulations using standard test systems for water and electric power distribution networks.

A. Water Distribution Network

Here, we use the standard 32–node Hanoi water distribution network, with model parameters taken from [25]. The network has 34 pipes and 3 cycles; nodes 1, 14, and 30 correspond to the source nodes, and the remaining nodes are the demand nodes. This test bed uses the Hazen-Williams head-loss model, in which $g(\theta_l-\theta_j) = \mathrm{sgn}(\theta_l-\theta_j)|\theta_l-\theta_j|^{\frac{1}{\beta}}$ and $\gamma_{lj} = 1/K_{lj}^{1/\beta}$ with $\beta = 1.85$ and K_{lj} given in (22) with the corresponding Hazen-Williams coefficients $b = 162.5$ and $c = 130$. We apply the entropy-like penalty-based approach from Section III and solve P2 for different values of ρ using the standard MATLAB solver *fmincon*; these values are compared with the global solution of P1. Table I contains the values of the cost function $F(u) := \sum_{i\in\mathcal{V}} F_i(u_i)$ with $F_i(u_i) = 0.01u_i^2$ for different values of ρ after 30 iterations. Convergence to the solution that satisfies the cycle constraints becomes faster as ρ increases. However, in order to keep the solution close to the global one, ρ needs to be small enough. To show that by solving P2 we satisfy the cycle constraints, $\sum_{(l,j)\in\vec{\mathcal{E}}_i} h\left(\frac{f_{lj}}{\gamma_{lj}}\right) = 0$ for all three cycles, in Table II we also provide

$$\epsilon := \max_{1\le i\le c}\Big|\sum_{(l,j)\in\vec{\mathcal{E}}_i} h\left(\frac{f_{lj}}{\gamma_{lj}}\right)\Big|, \tag{76}$$

for different values of ρ, which confirm our findings in Proposition 1, that is, $\epsilon[k] \to 0$, as (u, f) converges to the solution.

B. Electric Power Network

Here, we use the IEEE 39–bus power system, with all model parameters taken from [29]. In this example, the electrical lines are assumed to be lossless; thus, $g(x) = \sin(x)$, and voltage magnitudes are set to 1 pu at all buses. When we apply the entropy-like penalty-based approach, we obtain results similar to the ones from the previous example on the water distribution network. These results are stored in Table II, which contains the values of the cost function $F(u)$ and ϵ for different values of ρ after 150 iterations, where ϵ is very close to zero.

TABLE II: Values of the cost $F(u)$ and ϵ for the IEEE 39–bus power system.

ρ	$F(u)$ (p.u.)	ϵ (rad)
Global solution	85.804	0
1	85.815	8×10^{-4}
10	86.387	1.4×10^{-4}
100	94.198	2.6×10^{-5}
1000	103.13	0

TABLE III: Values of the cycle constraint violation level, $\epsilon_2[k]$, for $\mathcal{C}_2$, for the solutions of P3 and P4.

Solution of P3	$\epsilon_2[k] = -2.1$ deg
Solution of P4	$\epsilon_2[k] = 1.2 \times 10^{-5}$ deg

In the following, we apply the method based on solving P4 to a post-contingency flow problem where a certain transmission line (say a line connecting buses 4 and 5) opens and we need to determine the optimal power outputs for all ten generators. If we solve P3 instead of P4, its solution may not satisfy the cycle constraints. One of the main reasons is that the cost function $F(u)$ only takes into account local generation costs and ignores any stability constraints, which tend to be even more critical because of the lost electrical line. This also means that minimizing the generation cost does not guarantee that the power flows, f, resulting from the applied injections u, satisfy the cycle constraints in a post-contingency scenario. For illustration purposes, if the generation cost at bus 38 is much lower than that at buses 30, 37, and 39, then, the solution of P3 violates the cycle constraint along $\mathcal{C}_2 = \{2, 25, 26, 27, 17, 18, 3\}$, which is shown in Table III, where we give the values of $\epsilon_2[k]$, computed after finding the flows using an approach from Section IV. In contrast, solving P4 with the constraint in (70) yields a different set of injections that satisfy the cycle constraints, as shown in Table III.

VII. Concluding Remarks

In this paper, we proposed three approaches to solve the network flow problem under non-convex equality constraints on nodal variables along cycles. In the first one, we introduced an entropy-like penalty-based method to obtain a convex approximation of the original non-convex problem. In the second one, we solve the modified version of the original problem, where we drop the cycle constraints, and in order to recover the actual flows satisfying the cycle constraints, we solve a separate optimization problem, the solution of which maximizes the individual entropies of the cycles. In the third one, we obtained a convex inner approximation of the original non-convex constraint set for certain types of networks, which allowed us to recast the original non-convex problem as a convex problem.

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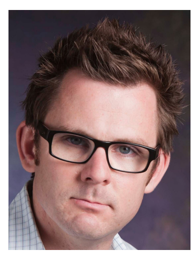

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