Power-Flow Formulation for Inverter-Based Grids

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***Abstract*—This paper presents a detailed formulation of the power-flow problem for inverter-based power systems. Specifically, we consider systems that comprise a group of grid-forming and grid-following inverter-based resources that are interconnected via a network of transmission lines. A full-order model of the power system is presented, and an associated steady-state model is derived, from which the corresponding power-flow problem is formulated. We provide numerical results comparing the steady-state response of the full-order models with the solution provided by the proposed power flow model.**

I. Introduction

Due to the increasing deployment of inverter-based resources (IBRs), and the progressive transition from synchronous generators to inverter-interfaced counterparts, the dynamical behavior of the electric power grid will be significantly altered. Accordingly, it is necessary to develop accurate mathematical models that capture the dynamic and steady-state characteristics of electrical power networks that rely exclusively on inverter-based resources for power generation.

The main contributions of this paper are as follows: (1) derive a steady-state model for power networks that are based on grid-forming (GFM) and grid-following (GFL) inverters, and (2) develop the power-flow formulation for such systems. For each bus that a GFM (GFL) inverter is connected to, we show that the power flow problem comprises five (three) algebraic equations that describe terminal relations between the inverter setpoints, internal state variables, the steady-state frequency of the grid, and the bus voltage magnitude and phase.

The development of full-order and reduced-order models for inverter-based power systems has received significant attention in the literature. The authors in [1], [2] present reduced-order models for microgrids that are obtained using small-signal analysis. The authors in [3]–[6] provide detailed results on the model-order reduction of inverter-based microgrids using singular perturbation analysis. The authors in [7], [8] develop a detailed full-order model for GFM inverter-based systems. The authors in [9], [10] present models for GFM inverter-based grids, which are obtained by performing successive model reduction steps on a full-order model using singular perturbation analysis. Although these efforts present reduced-order models for inverter-based power networks, a derivation of the corresponding steady-state power-flow formulations has not been derived; we address this gap.

Notation: The $n \times n$ diagonal matrix with diagonal entries $x_1, \dots, x_n$ is denoted by $\operatorname{diag}(x_1, \dots, x_n)$. The identity matrix is denoted by $\mathbb{I}$, the standard basis vector with 1 in the k-th entry is denoted by $\mathbb{e}_k$, and the all-zeros and all-ones vectors are denoted by $\mathbb{0}$ and $\mathbb{1}$, respectively. (Dimensions of $\mathbb{I}$, $\mathbb{e}_k$, $\mathbb{0}$, and $\mathbb{1}$ are not specified in the notation, with the understanding that they can be inferred from the context.)

Reference-frame Transformations: For balanced three-phase networks, we consider two direct-quadrature rotating reference frames as follows: (i) the DQ reference frame (also referred to as synchronously rotating reference frame), which rotates in synchrony with the system nominal angular frequency—assumed to be constant; and (ii) the dq reference frame, which rotates at a time-varying frequency whose value is determined by the reference angular frequency of a particular GFM inverter (see [11, pp. 69–114] for more details). In the remainder, all three-phase variables and companion signals are represented in either the DQ or dq reference frames. Denote

$$f'(t) = [f_D(t),\, f_Q(t)]^\top, \quad f''(t) = [f_d(t),\, f_q(t)]^\top, \quad (1)$$

as the DQ and dq representations of $f(t)$, respectively. Define

$$\delta(t) = \delta_0 + \int_0^t (\omega^\circ(x) - \omega_0)\, dx, \quad (2)$$

with $\delta_0 = \delta(0)$, and where ω_0 and $\omega^\circ(t)$ respectively denote the nominal angular frequency and the reference angular frequency of a particular inverter. Then, $f'(t)$ and $f''(t)$ are related via

$$f''(t) = \mathrm{R}(\delta(t)) f'(t), \quad (3)$$

where $\mathrm{R}(\cdot)$ is the transformation matrix given by

$$\mathrm{R}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Henceforth, we will simplify the adopted notation by dropping the time argument in all quantities.

Paper Organization: In Section II, we describe the steady-state models for the network, the GFM inverter, and the GFL inverter. In Section III, we formulate the power-flow problem for inverter-based power systems. In Section IV, numerical results that compare the response of the full-order model and the power-flow model are presented. Concluding remarks are provided in Section V.

II. The Steady-state Models

This section describes steady-state models for electrical power networks based on GFL and GFM inverters. The models are derived from full-order dynamic models of GFM and GFL inverters (see [12], [13] for detailed descriptions of such models) by setting the derivative of said models to zero.

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Consider the IBR connected to bus k of an electrical power network. Let ω_{ss} denote the steady-state frequency of the power network, and let ω_k denote the frequency of the IBR. In steady-state, we must have

$$\omega_k = \omega_{\mathrm{ss}}, \qquad \forall k. \tag{4a}$$

Also, let $s_{\mathrm{r},k}$ and $s_{\mathrm{r},0}$ denote the rated three-phase powers of the IBR and the electrical network, respectively, and define:

$$\sigma_k = \frac{s_{\mathrm{r},k}}{s_{\mathrm{r},0}}. \tag{4b}$$

Finally, let $i'_{\mathrm{g},k}$ denote the grid-side current of the IBR's LCL filter, in a per-unit system with base power $s_{\mathrm{r},k}$, and let i'_k denote the current injection at bus k, in a per-unit system with base power $s_{\mathrm{r},0}$ and same base voltage as $i'_{\mathrm{g},k}$. Then, it follows that:

$$i'_k = \sigma_k i'_{\mathrm{g},k}. \tag{4c}$$

A. The Power Network

We consider power networks comprising $n > 1$ buses that are interconnected via m short transmission lines.[1] Accordingly, each transmission line can be described using a series resistance and inductance, and without loss of generality, we assume there is at most one transmission line connecting each pair of buses.[2] We write $\mathcal{B} = \{1, 2, \ldots, n\}$ and $\mathcal{L} = \{1, 2, \ldots, m\}$ to denote the sets that index the buses and transmission lines in the power network, respectively. Suppose that *each bus of the power network has either a GFM inverter, a GFL inverter, or a constant power load connected to it.* We write $\mathcal{I}_{\mathrm{v}} \subseteq \mathcal{B}$ to denote the set of buses with a GFM inverter connected to them, and write $\mathcal{I}_{\mathrm{i}} \subseteq \mathcal{B}$ to denote the set of buses with a GFL inverter connected to them, so that $\mathcal{I}_{\mathrm{v}} \cap \mathcal{I}_{\mathrm{i}} = \emptyset$. Finally, we define $\mathcal{I} = \mathcal{I}_{\mathrm{v}} \cup \mathcal{I}_{\mathrm{i}}$ and write $\mathcal{N} = \mathcal{B} \setminus \mathcal{I}$ to denote the set of buses with a constant power load connected to them.

Let $\mathcal{E} \subset \mathcal{B} \times \mathcal{B} \setminus \{(k,k) : k \in \mathcal{V}\}$ denote the set of transmission lines so that $(k,j) \in \mathcal{E}$ if buses k and j are electrically connected, with the flow of power from bus k to bus j assigned to be positive. Let $\mathbb{L}$ denote a one-to-one mapping from $\mathcal{E}$ to $\mathcal{L} = \{1, 2, \ldots, m\}$ so that, for each $(k,j) \in \mathcal{E}$, there exists a unique $\ell \in \mathcal{L}$ that satisfies $\ell = \mathbb{L}(k,j)$. Accordingly, we can define a node-to-edge incidence matrix, $M = [m_{k\ell}] \in \mathbb{R}^{n \times m}$, with entries:

$$\begin{aligned} m_{k\ell} &= 1, && \text{if } \ell = \mathbb{L}(k,j), \quad (k,j) \in \mathcal{E}, \\ m_{k\ell} &= -1, && \text{if } \ell = \mathbb{L}(j,k), \quad (j,k) \in \mathcal{E}, \\ m_{k\ell} &= 0, && \text{otherwise.} \end{aligned}$$

The circuit model for each transmission line $\ell \in \mathcal{L}$ is the series connection of a resistance, r_ℓ, and an inductance, l_ℓ. In the DQ reference frame, let v'_k and i'_k denote the voltage and current injections at bus $i \in \mathcal{B}$, respectively, and let f'_ℓ denote the current flowing through transmission line $\ell = \mathbb{L}(k,j)$, all in per-unit. Then, the steady-state values of the line currents and the current injection at bus j are described by:

$$f'_\ell = \big(r_\ell \mathbb{I} - l_\ell \mathrm{R}(\tfrac{\pi}{2})\big)^{-1} \sum_{k \in \mathcal{B}} v'_k m_{k\ell}, \quad i'_k = \sum_{\ell=1}^{m} f'_\ell m_{k\ell}. \tag{5}$$

[1] A transmission line is typically categorized as *short* if its effective length is less than 50 miles (80 km) [14, p. 208].

[2] In the event that there are multiple transmission lines connecting a pair of buses, they can be collectively represented as a single transmission line.

TABLE I: Generic functions for droop, VSM, and dVOC.

	$f_{\mathrm{f},k}(x)$	$f_{\mathrm{v},k}(x)$	$f_{\mathrm{e},k}(x,y)$
droop/VSM	$d_{\mathrm{f},k}$	$d_{\mathrm{v},k}$	$x - y$
dVOC	$\frac{x^2}{\omega_0 \kappa_{1,k}}$	$\frac{x}{\kappa_{1,k}}$	$\kappa_{2,k}(x^2 - y^2)y$

B. The Grid-forming Inverter

1) Primary Controller: In per-unit, let $E_k^\star$ and E_k° denote the reference voltage-magnitudes obtained from a primary and a tertiary control scheme, respectively, and let $p_k^\star$ and $q_k^\star$ denote the reference active- and reactive-power injections obtained from secondary/tertiary control schemes, respectively. Let e'_k and v'_k denote the capacitor voltage of the inverter's LCL filter and the voltage at bus k, respectively. Let $p_{\mathrm{m},k}$ and $q_{\mathrm{m},k}$ denote low-pass filtered versions of the active- and reactive-power injections, respectively. Let η_k and α_k denote the internal state variable and output phase, respectively, of a phase-locked loop (PLL) that is used to compute the frequency of the bus voltage (the PLL is used in the VSM strategy, but not in droop or dVOC). In units of rad, let $\psi_k \in [0, 2\pi)$ denote a rotation angle that, in steady-state, determines the nature of the tradeoff between active power and reactive power in the voltage and frequency response of the GFM inverter. Then, the steady-state model for the primary controller is given by:

$$\omega_{\mathrm{ss}} - \omega_0 = \mathbb{e}_1^\top \mathrm{R}(\psi_k - \tfrac{\pi}{2}) \frac{1}{f_{\mathrm{f},k}(E_k^\circ)} \begin{bmatrix} p_k^\star - p_{\mathrm{m},k} \\ q_k^\star - q_{\mathrm{m},k} \end{bmatrix}, \tag{6a}$$

$$f_{\mathrm{e},k}(E_k^\circ, E_k^\star) = \mathbb{e}_2^\top \mathrm{R}(\psi_k - \tfrac{\pi}{2}) \frac{1}{f_{\mathrm{v},k}(E_k^\circ)} \begin{bmatrix} p_k^\star - p_{\mathrm{m},k} \\ q_k^\star - q_{\mathrm{m},k} \end{bmatrix}, \tag{6b}$$

$$\begin{bmatrix} p_{\mathrm{m},k} \\ q_{\mathrm{m},k} \end{bmatrix} = \frac{1}{\sigma_k} \begin{bmatrix} {e'_k}^\top i'_k \\ {e'_k}^\top \mathrm{R}(-\tfrac{\pi}{2}) i'_k \end{bmatrix}, \tag{6c}$$

$$0 = \mathbb{e}_2^\top \mathrm{R}(\alpha_k + \delta_k) v'_k, \qquad \eta_k = 0, \tag{6d}$$

where $f_{\mathrm{f},k}(\cdot)$, $f_{\mathrm{v},k}(\cdot)$, and $f_{\mathrm{e},k}(\cdot,\cdot)$ denote generic functions, whose values depend on the adopted primary-control strategy, i.e., droop, VSM, or dVOC, as detailed in Table I. In this table, $d_{\mathrm{f},k}$ and $d_{\mathrm{v},k}$ denote the frequency and voltage droop coefficients of a droop/VSM strategy, in $\mathrm{s\,rad}^{-1}$ and per-unit, respectively, and in per-unit, $\kappa_{1,k}$ and $\kappa_{2,k}$ denote the synchronization gain and the voltage-amplitude control gain of a dVOC strategy, respectively.

2) Current Limiter, PI Controllers, and LCL filter: Let ϕ''_k denote the internal state variable of a PI controller that regulates output voltage, and let $i''_{\mathrm{r},k}$ denote the output of the PI controller. Likewise, let γ''_k denote the internal state variable of a PI controller that regulates the inverter-side current, and let $u'_{\mathrm{r},k}$ denote the output of the PI controller. Let $i'_{\mathrm{i},k}$ and $i'_{\mathrm{g},k}$ denote the inverter- and grid-side currents of the LCL

filter, respectively. Then, the steady state model of the current limiter, the PI controller, and the filter are described by:

$$\begin{aligned} e'_k &= \big(\mathbb{I} + k_{\mathrm{b},k}(\rho_k - 1)A_{1,k}(\rho_k)\big)\mathrm{R}(-\delta_k)\mathbb{e}_1 E_k^{\circ} \\ &\quad - k_{\mathrm{b},k}(\rho_k - 1)A_{2,k}(\rho_k)v'_k, \end{aligned} \tag{6e}$$

$$i'_{\mathrm{i},k} = \rho_k \mathrm{R}(-\delta_k) i''_{\mathrm{r},k}, \tag{6f}$$

$$\gamma''_k = \frac{r_{\mathrm{i},k}}{k_{\mathrm{Ii},k}}\Big(\mathbb{I} + \frac{\omega_{\mathrm{ss}} - \omega_0}{\omega_0}\frac{l_{\mathrm{i},k}}{r_{\mathrm{i},k}}\mathrm{R}(\tfrac{\pi}{2})\Big)\mathrm{R}(\delta_k) i'_{\mathrm{i},k}, \tag{6g}$$

$$i''_{\mathrm{r},k} = A_{1,k}(\rho_k)\mathbb{e}_1 E_k^{\circ} - A_{2,k}(\rho_k)\mathrm{R}(\delta_k)v'_k, \tag{6h}$$

$$i'_k = \sigma_k\big(r_{\mathrm{g},k}\mathbb{I} - l_{\mathrm{g},k}\mathrm{R}(\tfrac{\pi}{2})\big)^{-1}(e'_k - v'_k), \tag{6i}$$

$$\begin{aligned} \phi''_k &= \frac{1}{k_{\mathrm{Iv},k}}\left(k_{\mathrm{Pv},k}k_{\mathrm{b},k}(\rho_k - 1) - \frac{\rho_k\omega_{\mathrm{ss}} - \omega_0}{\omega_0}\right) i''_{\mathrm{r},k} \\ &\quad + \frac{\omega_0 - \omega_{\mathrm{ss}}}{\omega_0\sigma_k}\mathrm{R}(\delta_k) i'_k, \end{aligned} \tag{6j}$$

$$u'_{\mathrm{r},k} = k_{\mathrm{Ii},k}\mathrm{R}(-\delta_k)\gamma''_k + e'_k - \rho_k\frac{\omega_{\mathrm{ss}} l_{\mathrm{i},k}}{\omega_0}\mathrm{R}(\tfrac{\pi}{2} - \delta_k) i''_{\mathrm{r},k}, \tag{6k}$$

$$\rho_k = -\,\varepsilon\ln\left(\exp\left(-\frac{1}{\varepsilon}\right) + \exp\left(-\frac{i_{\max}}{\varepsilon\|i''_{\mathrm{r},k}\|_2}\right)\right). \tag{6l}$$

where $k_{\mathrm{Iv},k}$ and $k_{\mathrm{Ii},k}$ denote the integral gains for PI controllers that regulate voltage and current, respectively, $k_{\mathrm{Pv},k}$ denotes the proportional gain of the PI controller that regulates voltage, $k_{\mathrm{b},k}$ denotes the integrator anti-windup gain of the PI controller that regulates voltage, ε denotes a small positive parameter which ensures a close approximation of ρ_k to the function $\min\left(1, \frac{i_{\max}}{\|i''_{\mathrm{r},k}\|_2}\right)$ (see [12] for details), $r_{\mathrm{i},k}$ and $l_{\mathrm{i},k}$ denote the inverter-side resistance and inductance of the filter, respectively, $r_{\mathrm{g},k}$ and $l_{\mathrm{g},k}$ denote sum of resistive and inductive elements (from the filter and transmission line) on the grid side, respectively, and c_k denotes the filter capacitance. Also, the matrices $\mathrm{A}_{1,k}(\cdot)$ and $\mathrm{A}_{2,k}(\cdot)$ are defined as follows:

$$\mathrm{A}_{1,k}(x) := \frac{1}{a_{5,k}(x)}\begin{bmatrix} a_{1,k}(x) & -a_{2,k}(x) \\ a_{2,k}(x) & a_{1,k}(x) \end{bmatrix}, \tag{7a}$$

$$\mathrm{A}_{2,k}(x) := \frac{1}{a_{5,k}(x)}\begin{bmatrix} a_{3,k}(x) & a_{4,k}(x) \\ -a_{4,k}(x) & a_{3,k}(x) \end{bmatrix}, \tag{7b}$$

where,

$$\begin{aligned} a_{1,k}(x) &= -\,(x-1)k_{\mathrm{b},k}\big((c_k l_{\mathrm{g},k} - 1)^2 + c_k^2 r_{\mathrm{g},k}^2\big) + x r_{\mathrm{g},k}, \\ a_{2,k}(x) &= x(c_k r_{\mathrm{g},k}^2 + (c_k l_{\mathrm{g},k} - 1)l_{\mathrm{g},k}), \\ a_{3,k}(x) &= (x-1)k_{\mathrm{b},k}(c_k l_{\mathrm{g},k} - 1) + x r_{\mathrm{g},k}, \\ a_{4,k}(x) &= \,-(x-1)k_{\mathrm{b},k}c_k r_{\mathrm{g},k} + x l_{\mathrm{g},k}, \\ a_{5,k}(x) &= (x-1)^2 k_{\mathrm{b},k}^2\big((c_k l_{\mathrm{g},k} - 1)^2 + (c_k r_{\mathrm{g},k})^2\big) \\ &\quad - 2x(x-1)k_{\mathrm{b},k} r_{\mathrm{g},k} + x^2(r_{\mathrm{g},k}^2 + l_{\mathrm{g},k}^2). \end{aligned}$$

C. The Grid-following Inverter

1) Phase-locked Loop: Let η_k denote the internal state variable of the PLL, and in the DQ reference frame, let e'_k denote the capacitor voltage of the LCL filter. Then, the steady-state model of the PLL is described by:

$$0 = \mathbb{e}_2^{\top}\mathrm{R}(\delta_k)e'_k, \qquad \eta_k = \frac{\omega_{\mathrm{ss}}}{\omega_0 k_{\mathrm{I}\theta,k}}, \tag{8a}$$

where $k_{\mathrm{I}\theta,k}$ denotes the integral gains of the PLL, in per-unit.

2) Current Limiter, PI Controller, and LCL filter: Leveraging the notation adopted in Section (II-B2), we have that the steady-state model of the GFL inverter's current-reference limiter, the PI controller, and the LCL filter are described by:

$$i'_{\mathrm{i},k} = \frac{\rho_k}{\|e'_k\|_2}\mathrm{R}(-\delta_k)\begin{bmatrix} p_k^{\star} \\ -q_k^{\star} \end{bmatrix}, \quad i''_{\mathrm{r},k} = \frac{1}{\|e'_k\|_2}\begin{bmatrix} p_k^{\star} \\ -q_k^{\star} \end{bmatrix}, \tag{8b}$$

$$\gamma''_k = \frac{r_{\mathrm{i},k}\rho_k}{k_{\mathrm{Ii},k}\|e'_k\|_2}\Big(\mathbb{I} + \frac{\omega_{\mathrm{ss}} - \omega_0}{\omega_0}\frac{l_{\mathrm{i},k}}{r_{\mathrm{i},k}}\mathrm{R}(\tfrac{\pi}{2})\Big)\begin{bmatrix} p_k^{\star} \\ -q_k^{\star} \end{bmatrix}, \tag{8c}$$

$$\begin{aligned} \mathbb{O} &= \Big(\big(r_{\mathrm{g},k}\mathbb{I} - l_{\mathrm{g},k}\mathrm{R}(\tfrac{\pi}{2})\big)^{-1} - c_k\mathrm{R}(\tfrac{\pi}{2})\Big)\mathbb{e}_1\|e'_k\|_2^2 - \big(r_{\mathrm{g},k}\mathbb{I} \\ &\quad - l_{\mathrm{g},k}\mathrm{R}(\tfrac{\pi}{2})\big)^{-1}\mathrm{R}(\delta_k)v'_k\|e'_k\|_2 - \rho_k\begin{bmatrix} p_k^{\star} \\ -q_k^{\star} \end{bmatrix}, \end{aligned} \tag{8d}$$

$$i'_k = \sigma_k\big(r_{\mathrm{g},k}\mathbb{I} - l_{\mathrm{g},k}\mathrm{R}(\tfrac{\pi}{2})\big)^{-1}(e'_k - v'_k), \tag{8e}$$

$$u'_{\mathrm{r},k} = k_{\mathrm{Ii},k}\mathrm{R}(-\delta_k)\gamma''_k + e'_k - \rho_k\frac{\omega_{\mathrm{ss}} l_{\mathrm{i},k}}{\omega_0\|e'_k\|_2}\mathrm{R}(\tfrac{\pi}{2} - \delta_k)\begin{bmatrix} p_k^{\star} \\ -q_k^{\star} \end{bmatrix}, \tag{8f}$$

$$\rho_k = -\varepsilon\ln\left(\exp\left(-\frac{1}{\varepsilon}\right) + \exp\left(-\frac{i_{\max}\|e'_k\|_2}{\varepsilon\sqrt{(p_k^{\star})^2 + (q_k^{\star})^2}}\right)\right). \tag{8g}$$

III. Formulation of the Power-flow Problem

In this section, we present the main results of this paper, namely: the power-flow problem formulation for power systems comprising groups of GFM inverters, GFL inverters, and loads that are interconnected via a network of short transmission lines.

A. The Power-flow Problem

Let $\boldsymbol{f}_{\mathrm{p},k}(\cdot)$ and $\boldsymbol{f}_{\mathrm{q},k}(\cdot)$ denote the active and reactive power injections, respectively, at bus $k \in \mathcal{B}$ of an electrical power network. Let $\boldsymbol{g}_{\mathrm{p},k}(\cdot)$ and $\boldsymbol{g}_{\mathrm{q},k}(\cdot)$ denote the active and reactive power injections, respectively, at terminals of an IBR connected to bus $k \in \mathcal{I}$. Let $\boldsymbol{h}_{1,k}(\cdot)$, $\boldsymbol{h}_{2,k}(\cdot)$, $\boldsymbol{h}_{3,k}(\cdot)$, $\boldsymbol{h}_{4,k}(\cdot)$, and $\boldsymbol{h}_{5,k}(\cdot)$ denote a set of functions that describe the inverter terminal relations for the GFM inverter connected to bus $k \in \mathcal{I}_{\mathrm{v}}$. Let $\boldsymbol{w}_{1,k}(\cdot)$, $\boldsymbol{w}_{2,k}(\cdot)$, and $\boldsymbol{w}_{3,k}(\cdot)$ denote a set of functions that describe the inverter terminal relations for the GFL inverter connected to bus $k \in \mathcal{I}_{\mathrm{i}}$. Let $-p_k^{\star}$ and $-q_k^{\star}$ denote the active and reactive power demand, respectively, at terminals of a constant power load connected to bus $k \in \mathcal{N}$.

The power-flow problem for electrical power networks based on IBRs is the computation of voltage magnitudes and phase angles at each bus of the network, while taking into account the steady-state terminal relations of the IBRs connected to those buses. Accordingly, the power-flow problem is described by the following system of equations:

$$-\,p_k^{\star} = \boldsymbol{f}_{\mathrm{p},k}(\cdot), \;\; -q_k^{\star} = \boldsymbol{f}_{\mathrm{q},k}(\cdot), \qquad \forall k \in \mathcal{N}, \tag{9a}$$

$$\boldsymbol{g}_{\mathrm{p},k}(\cdot) = \boldsymbol{f}_{\mathrm{p},k}(\cdot), \;\; \boldsymbol{g}_{\mathrm{q},k}(\cdot) = \boldsymbol{f}_{\mathrm{q},k}(\cdot), \qquad \forall k \in \mathcal{I}, \tag{9b}$$

$$\begin{aligned} 0 = \boldsymbol{h}_{1,k}(\cdot), \;\; 0 = \boldsymbol{h}_{2,k}(\cdot), \;\; 0 = \boldsymbol{h}_{3,k}(\cdot), \\ 0 = \boldsymbol{h}_{4,k}(\cdot), \;\; 0 = \boldsymbol{h}_{5,k}(\cdot), \quad \forall k \in \mathcal{I}_{\mathrm{v}}, \end{aligned} \tag{9c}$$

$$0 = \boldsymbol{w}_{1,k}(\cdot), \;\; 0 = \boldsymbol{w}_{2,k}(\cdot), \;\; 0 = \boldsymbol{w}_{3,k}(\cdot), \quad \forall k \in \mathcal{I}_{\mathrm{i}}. \tag{9d}$$

Next, we derive the expressions for the functions $\boldsymbol{f}_{\mathrm{p},k}(\cdot)$, $\boldsymbol{f}_{\mathrm{q},k}(\cdot)$, $\boldsymbol{g}_{\mathrm{p},k}(\cdot)$, $\boldsymbol{g}_{\mathrm{q},k}(\cdot)$, $\boldsymbol{h}_{1,k}(\cdot)$, $\boldsymbol{h}_{2,k}(\cdot)$, $\boldsymbol{h}_{3,k}(\cdot)$, $\boldsymbol{h}_{4,k}(\cdot)$, $\boldsymbol{h}_{5,k}(\cdot)$, $\boldsymbol{w}_{1,k}(\cdot)$, $\boldsymbol{w}_{2,k}(\cdot)$, and $\boldsymbol{w}_{3,k}(\cdot)$.

B. Power Injections at Network Buses

To derive expressions for the power injection at each bus of the electrical power network, define:

$$V_k := \|v_k'\|_2, \qquad \theta_k := \tan^{-1}\left(\frac{\mathbb{e}_2^\top v_k'}{\mathbb{e}_1^\top v_k'}\right). \tag{10}$$

Also, for $\ell = \mathbb{L}(k,j)$, define the *symmetric* matrices $G = [G_{kj}] \in \mathbb{R}^{b\times b}$ and $B = [B_{kj}] \in \mathbb{R}^{b\times b}$, with entries:

$$\begin{aligned}
&G_{kk} = -\sum_{j=1}^{n} G_{kj}, \quad && B_{kk} = -\sum_{j=1}^{n} B_{kj}, && \\
&G_{kj} = \frac{-r_\ell}{r_\ell^2 + l_\ell^2}, && B_{kj} = \frac{l_\ell}{r_\ell^2 + l_\ell^2}, && \text{if, } (k,j) \in \mathcal{E}, \\
&G_{kj} = 0, && B_{kj} = 0, && \text{if, } (k,j) \notin \mathcal{E}.
\end{aligned}$$

Using the steady-state model in (5) and the above definitions, we have that the net current injection at bus k of the electrical power network is given by:

$$i_k' = \sum_{j=1}^{n} \begin{bmatrix} G_{kj} & B_{kj} \\ B_{kj} & -G_{kj} \end{bmatrix} \begin{bmatrix} V_j\cos\theta_j - V_k\cos\theta_k \\ V_k\sin\theta_k - V_j\sin\theta_j \end{bmatrix}.$$

Accordingly, the active power injected into bus k of the electrical network is given by:

$$\begin{aligned}
V_k\mathbb{e}_1^\top \mathrm{R}(\theta_k) i_k' = V_k \sum_{j=1}^{n} V_j\big(&G_{kj}\cos(\theta_k-\theta_j) \\
&+ B_{kj}\sin(\theta_k-\theta_j)\big), \\
=: \boldsymbol{f}_{\mathrm{p},k}&\big(V_1,\theta_1,\dots,V_n,\theta_n\big),
\end{aligned} \tag{11a}$$

and the corresponding reactive power injection is given by

$$\begin{aligned}
V_k\mathbb{e}_1^\top \mathrm{R}(\theta_k - \tfrac{\pi}{2}) i_k' = V_k \sum_{j=1}^{n} V_j\big(&G_{kj}\sin(\theta_k-\theta_j) \\
&- B_{kj}\cos(\theta_k-\theta_j)\big), \\
=: \boldsymbol{f}_{\mathrm{q},k}&\big(V_1,\theta_1,\dots,V_n,\theta_n\big).
\end{aligned} \tag{11b}$$

C. Power Injections at Inverter Terminals

To derive expressions for power injections at the terminals of each IBR connected to the electrical power network, define:

$$E_k := \|e_k'\|_2, \qquad \vartheta_k := \tan^{-1}\left(\frac{\mathbb{e}_2^\top e_k'}{\mathbb{e}_1^\top e_k'}\right). \tag{12}$$

From (12), (6i), and (8e), we have that the net current injection at the terminals of a GFL/GFM inverter is given by

$$i_k' = \sigma_k\big(r_{\mathrm{g},k}\mathbb{I} - l_{\mathrm{g},k}\mathrm{R}(\tfrac{\pi}{2})\big)^{-1}(\mathrm{R}(-\vartheta_k)\mathbb{e}_1 E_k - \mathrm{R}(-\theta_k)\mathbb{e}_1 V_k).$$

Accordingly, the active power flow from the IBR terminals to bus k is given by:

$$\begin{aligned}
V_k\mathbb{e}_1^\top \mathrm{R}(\theta_k) i_k' = \frac{-\sigma_k}{r_{\mathrm{g},k}^2 + l_{\mathrm{g},k}^2}\Big(&V_k^2 r_{\mathrm{g},k} \\
&- V_k E_k\big(r_{\mathrm{g},k}\cos(\vartheta_k-\theta_k) + l_{\mathrm{g},k}\sin(\vartheta_k-\theta_k)\big)\Big) \\
=: \boldsymbol{g}_{\mathrm{p},k}&\big(V_k, E_k, \theta_k, \vartheta_k\big),
\end{aligned} \tag{13a}$$

and the corresponding reactive power flow from the IBR terminals to bus k is given by:

$$\begin{aligned}
V_k\mathbb{e}_1^\top \mathrm{R}(\theta_k - \tfrac{\pi}{2}) i_k' = \frac{-\sigma_k}{r_{\mathrm{g},k}^2 + l_{\mathrm{g},k}^2}\Big(&V_k^2 l_{\mathrm{g},k} \\
&+ V_k E_k\big(r_{\mathrm{g},k}\sin(\vartheta_k-\theta_k) \\
&- l_{\mathrm{g},k}\cos(\vartheta_k-\theta_k)\big)\Big) \\
=: \boldsymbol{g}_{\mathrm{q},k}&\big(V_k, E_k, \theta_k, \vartheta_k\big).
\end{aligned} \tag{13b}$$

D. Inverter Terminal Relations

Consider the IBR connected to bus k of an electrical power network. The steady-state terminal relations of the IBR can be described by the models presented in Sections II-B and II-C.

1) The Grid-forming Inverter: The steady-state relations between the GFM inverter state variables, i.e., ω_{ss}, ρ_k, E_k°, δ_k, E_k, ϑ_k, and the variables associated with bus k, i.e., V_k and θ_k are defined as follows:

$$\begin{aligned}
0 = \Big(&p_k^\star - \frac{1}{r_{\mathrm{g},k}^2 + l_{\mathrm{g},k}^2}\big(E_k^2 r_{\mathrm{g},k} - E_k V_k\big(r_{\mathrm{g},k}\cos(\vartheta_k-\theta_k) \\
&- l_{\mathrm{g},k}\sin(\vartheta_k-\theta_k)\big)\big)\Big)\frac{\sin\psi_k}{f_{\mathrm{f},k}(E_k^\circ)} - \Big(q_k^\star \\
&- \frac{1}{r_{\mathrm{g},k}^2 + l_{\mathrm{g},k}^2}\big(E_k^2 l_{\mathrm{g},k} - E_k V_k\big(r_{\mathrm{g},k}\sin(\vartheta_k-\delta_k) \\
&+ l_{\mathrm{g},k}\cos(\vartheta_k-\delta_k)\big)\big)\Big)\frac{\cos\psi_k}{f_{\mathrm{f},k}(E_k^\circ)} - \omega_{\mathrm{ss}} + \omega_0 \\
=: \boldsymbol{h}_{1,k}&\big(\omega_{\mathrm{ss}}, E_k^\circ, V_k, E_k, \theta_k, \vartheta_k\big),
\end{aligned} \tag{14a}$$

$$\begin{aligned}
0 = \Big(&p_k^\star - \frac{1}{r_{\mathrm{g},k}^2 + l_{\mathrm{g},k}^2}\big(E_k^2 r_{\mathrm{g},k} - E_k V_k\big(r_{\mathrm{g},k}\cos(\vartheta_k-\theta_k) \\
&- l_{\mathrm{g},k}\sin(\vartheta_k-\theta_k)\big)\big)\Big)\frac{\cos\psi_k}{f_{\mathrm{v},k}(E_k^\circ)} + \Big(q_k^\star \\
&- \frac{1}{r_{\mathrm{g},k}^2 + l_{\mathrm{g},k}^2}\big(E_k^2 l_{\mathrm{g},k} - E_k V_k\big(r_{\mathrm{g},k}\sin(\vartheta_k-\delta_k) \\
&+ l_{\mathrm{g},k}\cos(\vartheta_k-\delta_k)\big)\big)\Big)\frac{\sin\psi_k}{f_{\mathrm{v},k}(E_k^\circ)} + f_{\mathrm{e},k}(E_k^\star, E_k^\circ) \\
=: \boldsymbol{h}_{2,k}&\big(E_k^\circ, V_k, E_k, \theta_k, \vartheta_k\big),
\end{aligned} \tag{14b}$$

$$\begin{aligned}
0 = &-E_k\cos\vartheta_k - V_k(\rho_k - 1)k_{\mathrm{b},k}\mathbb{e}_1^\top A_{2,k}(\rho_k)\mathrm{R}(-\theta_k)\mathbb{e}_1 \\
&+ E_k^\circ\big(\cos\delta_k + (\rho_k-1)k_{\mathrm{b},k}\mathbb{e}_1^\top A_{1,k}(\rho_k)\mathrm{R}(-\delta_k)\mathbb{e}_1\big) \\
=: &\,\boldsymbol{h}_{3,k}\big(\rho_k, E_k^\circ, \delta_k, V_k, E_k, \theta_k, \vartheta_k\big),
\end{aligned} \tag{14c}$$

$$\begin{aligned}
0 = &-E_k\sin\vartheta_k - V_k(\rho_k - 1)k_{\mathrm{b},k}\mathbb{e}_2^\top A_{2,k}(\rho_k)\mathrm{R}(-\theta_k)\mathbb{e}_1 \\
&+ E_k^\circ\big(\sin\delta_k + (\rho_k-1)k_{\mathrm{b},k}\mathbb{e}_2^\top A_{1,k}(\rho_k)\mathrm{R}(-\delta_k)\mathbb{e}_1\big) \\
=: &\,\boldsymbol{h}_{4,k}\big(\rho_k, E_k^\circ, \delta_k, V_k, E_k, \theta_k, \vartheta_k\big),
\end{aligned} \tag{14d}$$

$$\begin{aligned}
0 = \rho_k + \varepsilon\ln\Bigg(&\exp\Bigg(-i_{\max} \div \varepsilon\|A_{1,k}(\rho_k)\mathbb{e}_1 E_k^\circ \\
&- A_{2,k}(\rho_k)\mathrm{R}(\delta_k-\theta_k)\mathbb{e}_1 V_k\|_2\Bigg) + \exp\left(\frac{-1}{\varepsilon}\right)\Bigg) \\
=: \boldsymbol{h}_{5,k}&\big(\rho_k, E_k^\circ, \delta_k, V_k, \theta_k\big).
\end{aligned} \tag{14e}$$

2) The Grid-following Inverter: The steady-state relations between the GFL inverter state variables, i.e., ρ_k, E_k, ϑ_k, and the variables associated with bus k, i.e., V_k and θ_k are defined as follows:

$$\begin{aligned}0 &= -\rho_k p_k^\star + \frac{1}{r_{\mathrm{g},k}^2 + l_{\mathrm{g},k}^2}\big(E_k^2 r_{\mathrm{g},k} - E_k V_k \big(r_{\mathrm{g},k}\cos(\vartheta_k - \theta_k) \\ &\quad - l_{\mathrm{g},k}\sin(\vartheta_k - \theta_k)\big)\big) \\ &=: \boldsymbol{w}_{1,k}\big(\rho_k, V_k, E_k, \theta_k, \vartheta_k\big), \end{aligned} \tag{15a}$$

$$\begin{aligned}0 &= -\rho_k q_k^\star + \frac{1}{r_{\mathrm{g},k}^2 + l_{\mathrm{g},k}^2}\big(E_k^2 l_{\mathrm{g},k} - E_k V_k \big(r_{\mathrm{g},k}\sin(\vartheta_k - \theta_k) \\ &\quad + l_{\mathrm{g},k}\cos(\vartheta_k - \theta_k)\big)\big) - c_k E_k^2 \\ &=: \boldsymbol{w}_{2,k}\big(\rho_k, V_k, E_k, \theta_k, \vartheta_k\big), \end{aligned} \tag{15b}$$

$$\begin{aligned}0 &= \rho_k + \varepsilon \ln\left(\exp\left(\frac{-1}{\varepsilon}\right) + \exp\left(\frac{-i_{\max}E_k}{\varepsilon\sqrt{(p_k^\star)^2 + (q_k^\star)^2}}\right)\right) \\ &=: \boldsymbol{w}_{3,k}\big(\rho_k, E_k\big). \end{aligned} \tag{15c}$$

Remark. *The system of equations in* (9)*, and the corresponding definitions provided in in* (11)*,* (13)*,* (14)*, and* (15) *reveal that, for power networks with IBRs connected to each bus, the power flow problem is described by a total of* $2|\mathcal{N}| + 2|\mathcal{I}| + 5|\mathcal{I}_{\mathrm{v}}| + 3|\mathcal{I}_{\mathrm{i}}|$ *algebraic equations. Choosing* $\theta_1 = 0$ *as the reference angle for all voltage phases, we have that the variables to be solved for in the power-flow problem are described as follows: (1)* ω_{ss}, $\theta_2, \ldots, \theta_{|\mathcal{B}|}$*, and* $V_1, \ldots, V_{|\mathcal{B}|}$*, (2) For all* $k \in \mathcal{I}_{\mathrm{v}}$ *and* $k \in \mathcal{I}_{\mathrm{i}}$*, the unknowns are:* ϑ_k, E_k, ρ_k*, (3) Two additional unknowns for buses indexed by* $k \in \mathcal{I}_{\mathrm{v}}$ *are* E_k° *and* δ_k*. Since the number of equations match the number of unknowns, we can solve for the unknowns.*

IV. Numerical Results

In this section, we present numerical results that validate the power-flow models developed in this paper. A 3-bus network comprising two GFM inverters and one GFL inverter, which are interconnected via short transmission lines, is modeled using the full-order model as well as the power-flow equations and their supporting equations. The steady-state model is used to compute steady-state initial conditions for both models, and the same disturbances are implemented in both models. The parameters for the 3-bus network are obtained from [13].

We introduce disturbances into the system via $p_1^\star$ and $q_1^\star$. At $t = 0.5$ s and $t = 1$ s, the value of $p_1^\star$ is sequentially decreased by 0.1 pu, and at $t = 2$ s, the value $q_1^\star$ is decreased by 0.1 pu. The evolution of the bus frequency, the steady-state frequency from the power-flow solution, and the voltage magnitude of each bus are depicted in Fig. 2. The results demonstrates that after a disturbance is introduced into both models, the solution of the steady-state model is able to track the steady-state values of corresponding states in the full-order model.

V. Concluding Remarks & Future Work

This work developed the steady-state model and the power-flow formulation for inverter-based power systems. Compared to previous efforts for model reduction, our models describe the steady-state characteristics of such networks and captures

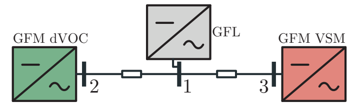

Fig. 1: The 3-bus network with a GFL IBR, a dVOC GFM IBR, and a VSM GFM IBR connected to buses 1, 2, and 3, respectively.

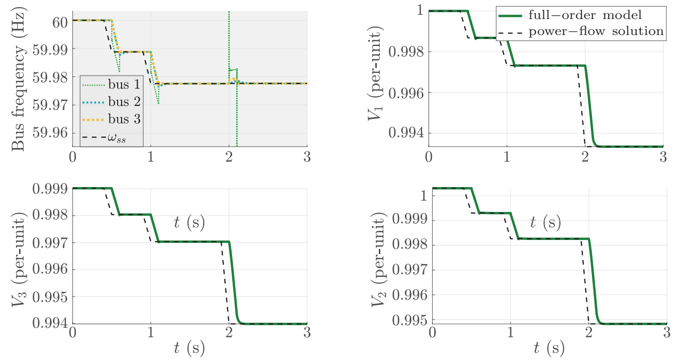

Fig. 2: Results of the steady-state and dynamic simulations.

the steady-state behavior of the system frequency in the presence of disturbances. Simulation results are presented to validate the response of the model under various disturbances.

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