An Online Feedback Optimization Approach to Voltage Regulation in Inverter-Based Power Distribution Networks

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***Abstract*— We address the problem of controlling the reactive power setpoints of a set of distributed energy resources (DERs) in a power distribution network so as to mitigate the impact of variability in uncontrolled power injections associated with, e.g., renewable-based generation. We formulate the control design problem as a stochastic optimization problem, which we solve online using a modified version of a projected stochastic gradient descent (PSGD) algorithm. The proposed PSGD-based algorithm utilizes sensitivities of changes in bus voltage magnitudes to changes in DER reactive power setpoints; such sensitivities are learned online via a recursive least squares estimator (rLSE). To ensure proper operation of the rLSE, the sequence of incremental changes in DER reactive power setpoints needs to be persistently exciting, which is guaranteed by a mechanism built into the controller. We analyze the stability of the closed-loop system and showcase controller performance via numerical simulations on the IEEE 123-bus distribution test feeder.**

I. Introduction

Currently, voltage regulation in power distribution networks is accomplished, for the most part, through tap-changing under-load transformers (TCULs) and fixed/switched capacitor banks. However, while these devices are effective in managing slow changes in voltage (minutes to hours), they are not suitable for managing fast voltage fluctuations (seconds to minutes) arising, e.g., from rapid changes in renewable-based power generation. This problem can be effectively addressed by controlling the reactive power injected into the distribution network by power-electronic inverter-interfaced DERs—a solution that has been actively pursued in the last decade (see, e.g, [1], [2] and the references therein).

In this paper, we also pursue the idea of utilizing inverter-interfaced DERs for voltage regulation. Building on our earlier work on data-driven control algorithms in [2], [3], [4], we design a voltage regulation scheme that does not rely on an a priori known model of the system to be controlled; instead, the scheme utilizes data to estimate such a model online, while simultaneosly executing a feedback control algorithm. We demonstrate that the proposed approach is adaptive to constantly varying system conditions and disturbances, and is capable of utilizing the reactive power support capabilities of inverter-based resources to effectively provide voltage regulation at a fast time-scale (i.e., milliseconds to seconds).

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The control design problem is cast as a stochastic optimization problem whose goal is to determine the reactive power setpoints of DERs so as to minimize the expectation of bus voltage deviations from their nominal values. In order to solve this problem, we utilize a projected stochastic gradient descent (PGSD) algorithm (see e.g., [5]). Proper initialization and execution of the algorithm for a single step essentially results in a feedback controller that utilizes measured voltage deviations to adjust reactive power setpoints of DERs. In its basic form, the algorithm relies on knowing the sensitivities of changes in bus voltages with respect to changes in reactive power injections. Instead of obtaining these sensitivities offline via a model, we design a recursive least squares estimator (rLSE) that learns them online by using real-time voltage measurement data and the sequence of reactive power setpoints generated by the controller. The rLSE is executed in parallel with the controller, and in order to ensure proper operation of the estimator, the sequence of incremental changes in DER reactive power setpoints needs to be persistently exciting. To this end, we modify the basic control algorithm to include a mechanism that ensures this.

Most of the existing literature on utilization of inverter-interfaced DERs for voltage regulation rely on the use of exact models of the network (see, e.g., [1] and the references therein). However, these methodologies have practical limitations in that, due to the limited number of sensors in power distribution networks, the models are hard to obtain in practice. More recently, there have been several papers that propose the use of data-driven techniques for addressing the voltage regulation problem in power distribution networks, as well as in other related control and estimation problems. For example, the authors of [3] present a data-driven framework for coordinating the active and reactive power injections of DERs to provide voltage regulation in radial networks. The proposed scheme utilizes estimates of network voltage sensitivities obtained online by fitting measurements to the so-called LinDistFlow model; by contrast, the method we propose here does not assume any particular model structure. The authors of [2] propose a data-driven voltage regulation approach based on the estimated topology and line parameters of radial power distribution systems. The authors of [4] propose a model-free control scheme for regulating voltage, frequency, and line flows that assumes no prior information on the system. The use of linear estimation techniques to learn sensitivities of various power system state variables to control inputs has also been exploited in numerous applications in bulk power system monitoring and control [6], [7], [8].

The framework proposed in this paper is closely related to that presented in [9], which proposes a model-free, real-time optimal power flow solver through feedback optimization. Similar to the approach we adopt here, [9] proposes a data-driven online approach to learn a model of input–output sensitivities. The objective function of the online feedback optimization formulated is the operational cost of the inputs, and the inequality constraints are the operational limitations on the inputs. The sensitivity estimate at each iteration is computed using both the previous estimate and the current measurements only, and a projected gradient solution method for the optimization problem, with persistent excitation of the control inputs performed at each iteration. By contrast, in our work, we formulate the control problem as a stochastic optimization problem, where the objective function is the expected value of the norm of the deviation of the outputs from their nominal values. In addition, the sensitivity estimate in our work is computed using all previous measurements with less weights assigned to older measurements. Finally, the proposed scheme in [9] does not account for the constraints on control inputs when attempting to generate persistently exciting inputs and does not have a built-in mechanism to check whether or not the sequence of control inputs is indeed persistently exciting. By contrast, our work does take these important issues into account.

II. Problem Formulation

Consider a power distribution network with $n+1$ buses indexed by the elements in $\mathcal{N} = \{0, 1, 2, \ldots, n\}$, where the 0 element corresponds to the bus at which the network is connected to an external system, e.g., a sub-transmission grid. Assume the network has m reactive-power-capable DERs indexed by the elements in $\mathcal{C} = \{1, 2 \ldots, m\}$. Let $v_i(t)$ denote the magnitude of the phasor associated with the voltage at bus $i,\ i \in \mathcal{N}$, at time t. Assume that $v_0(t) = V_0$, where V_0 is a positive constant, for all $t \geq 0$, and define $v(t) := \big[v_1(t), v_2(t), \ldots, v_n(t)\big]^\top$. Also, let $q_i(t)$ denote the reactive power injected into the network by reactive-power capable DER $i,\ i \in \mathcal{C} = \{1, 2 \ldots, m\}$, at time t, and define $q(t) := \big[q_1(t), q_2(t), \ldots, q_m(t)\big]^\top$.

Assume that, initially, $q(t_0) = q_0$, where $q_0 \in \mathbb{R}^m$ is given. Then, the objective is to adjust $q(t_k) =: q_k,\ k \geq 1$, via a feedback controller so as to regulate $v(t)$ to some $v^* = \big[v_1^*, v_2^*, \ldots, v_n^*\big]^\top$, where $v_i^* > 0,\ i = 1, 2, \ldots, n$, denotes some nominal value. Let $\{t_k^+\}_{k\geq 1}$, $t_k < t_k^+ < t_{k+1}$, denote the sequence of time instants at which $v(t)$ is measured, and define $v_k := v(t_k^+)$. Then, we have that

$$v_k = h\big(q_k, w_k\big), \tag{1}$$

where $w_k = \big[w_{1,k}, w_{2,k}, \ldots, w_{d,k}\big]^\top$, with $w_{j,k} := w_j(t_k),\ j = 1, 2, \ldots, d$, representing exogenous disturbances associated with, e.g., DER active power generation, load active and reactive power demand, and network parameters.

Remark 1: In general, it is difficult to analytically characterize the function $h(\cdot, \cdot)$ as this essentially entails obtaining an analytical solution to the power flow problem, which is defined by a set of nonlinear equations describing the active and reactive power balance at each bus (see, e.g., [10, pp. 323-330]). To be more specific, these equations map the magnitudes and phase angles of the phasors associated with bus voltages to the active and reactive power injections at all buses; hence one would need to invert such mapping to analytically characterize $h(\cdot, \cdot)$. Therefore, in the remainder we will assume that $h(\cdot, \cdot)$ is not known.

Assume that values taken by $w_k,\ k \geq 1$, are of the form

$$w_k = w_k^\circ + \xi_k, \tag{2}$$

with the value taken by w_k° known and slowly changing as k evolves, and where ξ_k is not a priori known but can be described by a random vector $\Xi_k \in \mathbb{R}^d$. In the remainder, we assume that the random vectors $\Xi_k,\ k \geq 0$, are independent and identically distributed (i.i.d.) with zero mean, i.e.,

$$\mathrm{E}\big[\Xi_k\big] = 0, \quad k \geq 1, \tag{3}$$

and whose covariance matrix is such that

$$\mathrm{E}\big[\|\Xi_k\|_2^2\big] \leq \sigma, \tag{4}$$

where σ denotes some positive constant. Now, since the values taken by $w_k^\circ,\ k \geq 1$, are known, we can equivalently describe the relation in (1) as follows:

$$v_k = h_k(q_k, \xi_k), \tag{5}$$

where $h_k(q_k, \xi_k) := h(q_k, w_k^\circ + \xi_k)$. In the remainder, we will assume q_0 is such that

$$\begin{aligned} v_0 &= h(q_0, w_0^\circ) \\ &= v^*, \end{aligned} \tag{6}$$

with the value taken by w_0° known, i.e., initially there is no uncertainty in the value that the exogenous disturbances takes, and the reactive powers injected by the DERs are such that the magnitudes of all bus voltages are equal to their nominal values.

III. Model-Based Feedback Optimization

In order to determine the value of $q_k,\ k \geq 1$, consider the following optimization problem:

$$\underset{\varphi_k}{\text{minimize}} \quad \frac{1}{2}\mathrm{E}_{\Xi_k}\Big[\big\|v^* - h_k\big(\varphi_k, \Xi_k\big)\big\|_2^2\Big] \tag{7a}$$

$$\text{subject to} \quad \underline{q}_k \leq \varphi \leq \overline{q}_k, \tag{7b}$$

where $\mathrm{E}_{\Xi_k}[\cdot]$ denotes expectation over the distribution of Ξ_k, and $\underline{q}_k$ and $\overline{q}_k$ denote the lower and upper bounds, respectively, on the reactive power that can be injected by DERs into the network. Then, a local minimum of (7), which we denote by φ_k^*, can be approximately obtained as $\varphi_k^* \approx \lim_{r\to\infty} \varphi_k^r$, with the evolution of φ_k^r, governed by

$$\varphi_k^{r+1} = \left[\varphi_k^r + \gamma_k \left(\left.\frac{\partial h_k(\varphi, \xi)}{\partial \varphi}\right|_{\varphi=\varphi_k^r,\ \xi=\xi_k^r}\right)^\top \times \Big(v^* - h_k\big(\varphi_k^r, \xi_k^r\big)\Big)\right]_{\underline{q}_k}^{\overline{q}_k}, \quad r \geq 0, \tag{8}$$

where γ_k is a positive constant, and $\xi_k^r,\ r \geq 0$, denote samples from the distribution of Ξ_k. The algorithm in (8) is essentially a PSGD algorithm for the problem in (7) (see, e.g., [5] and the references therein).

We initialize the PSGD algorithm in (8) as follows

$$\begin{aligned} \varphi_k^0 &= q_{k-1}, \\ \xi_k^0 &= \xi_{k-1}, \end{aligned} \tag{9}$$

where ξ_{k-1} denotes the realized value of Ξ_{k-1}, and instead of running it to completion, we will only execute one iteration and set $q_k = \varphi_k^1$. Then, it follows that the evolution of q_k is governed by

$$\begin{aligned} q_k = \Bigg[q_{k-1} + \gamma_k \left(\left. \frac{\partial h_k(\varphi,\xi)}{\partial \varphi} \right|_{\varphi = q_{k-1},\ \xi = \xi_{k-1}} \right)^\top \\ \times \Big(v^* - h_k\big(q_{k-1}, \xi_{k-1}\big) \Big) \Bigg]_{\underline{q}_k}^{\overline{q}_k}. \end{aligned} \tag{10}$$

The one-step PSGD-based algorithm proposed in (10) for solving the problem in (7) is a special case of the setting in [11], which proposes a framework to sequentially solve stochastic minimization problems with slowly varying cost functions that are convex.

Remark 2: We assume for subsequent developments that the problem (7) changes slowly with k, while the one-step PSGD-based algorithm is executed, which entails assuming that the $h_k(\cdot,\cdot)$'s, $\underline{q}_k$ and $\overline{q}_k$ vary slowly with k. However, the algorithm will also work in the presence of large but infrequent variations due to, e.g., weather changes, as long as there are sufficiently long time intervals during which the algorithm is able to converge.

Note that by using ξ_{k-1} to initialize the PSGD-based algorithm, we are using a sample from Ξ_{k-1}; however, recall that we have assumed that the Ξ_l's are i.i.d., so we are effectively sampling from the distribution of Ξ_k. Then, since we have assumed that $h_k(\cdot,\cdot)$ changes slowly with k, we can make the following approximations:

$$\begin{aligned} h_k\big(q_{k-1}, \xi_{k-1}\big) &\approx h_{k-1}\big(q_{k-1}, \xi_{k-1}\big) \\ &= v_{k-1}, \end{aligned} \tag{11}$$

$$\left. \frac{\partial h_k(\varphi,\xi)}{\partial \varphi} \right|_{\varphi = q_{k-1},\ \xi = \xi_{k-1}} \approx \left. \frac{\partial h_{k-1}(\varphi,\xi)}{\partial \varphi} \right|_{\varphi = q_{k-1},\ \xi = \xi_{k-1}}. \tag{12}$$

Thus, if we assume measurements of $v_k,\ k \geq 1$, are available, we can update the value of q_k as follows:

$$q_k = \Big[q_{k-1} + \gamma_k S_{k-1}^\top \big(v^* - v_{k-1}\big) \Big]_{\underline{q}_k}^{\overline{q}_k}, \tag{13}$$

where

$$S_{k-1} = \left. \frac{\partial h_{k-1}(\varphi,\xi)}{\partial \varphi} \right|_{\varphi = q_{k-1},\ \xi = \xi_{k-1}} \in \mathbb{R}^{n \times m}. \tag{14}$$

Note that in order to execute (13), we would need to compute $\partial h_{k-1}(\varphi,\xi)/\partial\varphi$ for $\varphi = q_{k-1}$ and $\xi = \xi_{k-1}$. This computation can be done by manipulating the power flow Jacobian without necessarily solving the power flow equations; however, it requires knowing the value of ξ_{k-1}, which we have assumed it is not a priori known. To circumvent this issue, instead of using S_{k-1} in (13), we will use an estimate obtained using measurements of $\big\{q_l, v_l\big\}_{l=0}^{k-1}$; the construction of such an estimate is detailed next.

IV. Online Matrix Sensitivity Estimator

Define $\Delta v_k := v_k - v_{k-1}$, $\Delta q_k := q_k - q_{k-1}$ and $\Delta\xi_k := \xi_k - \xi_{k-1}$; then, by using (5), we have that

$$v_{k-1} + \Delta v_k = h_k\big(q_{k-1} + \Delta q_k, \xi_{k-1} + \Delta\xi_k\big), \tag{15}$$

Now, by expanding the right-hand side of (15) about $\big(q_{k-1}, \xi_{k-1}\big)$ using the Taylor series expansion, and using the approximation in (12), it follows that

$$\Delta v_k = S_{k-1}\Delta q_k + \epsilon_k, \tag{16}$$

where ϵ_k represents higher order terms in Δq_k, all the terms in $\Delta\xi_k$, and the error associated with the use of (12). Then, by assuming that ϵ_k is much smaller that $S_{k-1}\Delta q_k$, we have that

$$\Delta v_k \approx S_{k-1}\Delta q_k. \tag{17}$$

Now let $\widehat{S}_k^*$ denote an estimate of S_k, which we can obtain using the method of least squares as follows:

$$\widehat{S}_k^* = \underset{S}{\operatorname{argmin}}\ \ell(S), \tag{18}$$

where

$$\ell(S) := \sum_{l=1}^{k} \lambda^{k-l} \big\| \Delta v_l - S\Delta q_l \big\|_2^2,$$

with $\lambda \in (0,1)$ denoting the forgetting factor that allows the estimator to assign exponentially less weight to older measurements and adapt to changes in operating conditions. The solution to the least-squares problem (18) can be derived as follows. We first compute the gradient of the loss function $\ell(S)$ in (18), which yields the following expression:

$$\nabla\ell(S) = -2\sum_{l=1}^{k} \lambda^{k-l} \big(\Delta v_l - S\Delta q_l\big)\Delta q_l^\top. \tag{19}$$

Then, we equate the gradient (19) to zero and obtain

$$\sum_{l=1}^{k} \lambda^{k-l} \Delta v_l \Delta q_l^\top = S\left(\sum_{l=1}^{k} \lambda^{k-l} \Delta q_l \Delta q_l^\top\right). \tag{20}$$

Now in order to solve for S in (20), we need to invert the matrix that multiplies S. Since this matrix is the sum of k $(m \times m)$-dimensional rank-one matrices, sufficient conditions for ensuring its invertibility are that: i) $k \geq m$, and ii) the sequence $\{\Delta q_l\}_{l=1}^k$ is persistently exciting.[1] Assuming these

[1] A discrete-time signal $x[t]$, $t = 1, 2, \ldots$, is persistently exciting if, for every k, there exist an integer l and constants $\varrho_1, \varrho_2 > 0$ such that the matrices $\varrho_1 I - \sum_{t=k}^{k+l} x[t]x[t]^\top$ and $\sum_{t=k}^{k+l} x[t]x[t]^\top - \varrho_2 I$ are positive definite [12].

two conditions are satisfied, we can now solve for S in (20) and obtain $\widehat{S}_k^*,\ k \geq m$:

$$\widehat{S}_k^* = \left(\sum_{l=1}^{k} \lambda^{k-l} \Delta v_l \Delta q_l^\top\right)\left(\sum_{l=1}^{k} \lambda^{k-l} \Delta q_l \Delta q_l^\top\right)^{-1}. \quad (21)$$

The issue of ensuring that the sequence $\{\Delta q_l\}_{l=1}^k$ is persistently exciting is addressed in the next section. However, even if this can be addressed satisfactorily, the estimate $\widehat{S}_k^*$ obtained using (21) is only valid for $k \geq m$; this means that in practice we would have to wait for m steps before we can obtain our first estimate. To address this, we use an algorithm that will recursively generate a sequence $\left\{\widehat{S}_k\right\}_{k\geq 1}$ that can be shown to converge to $\widehat{S}_k^*$ for k large enough. The update rules for this recursive algorithm are as follows:

$$\begin{aligned} \widehat{S}_k &= \widehat{S}_{k-1} + \Big(\Delta v_k - \widehat{S}_{k-1}\Delta q_k\Big)\Delta q_k^\top F_k, \\ F_k &= \lambda^{-1} F_{k-1} \\ &\quad - \frac{\lambda^{-2}}{1+\lambda^{-1}\Delta q_k^\top F_{k-1}\Delta q_k} F_{k-1}\Delta q_k \Delta q_k^\top F_{k-1}, \end{aligned} \quad (22)$$

with

$$\begin{aligned} \widehat{S}_0 &= \left.\frac{\partial h(\varphi, w)}{\partial \varphi}\right|_{\varphi=q_0,\ w=w_0^\circ}, \\ F_0 &= I_m, \end{aligned} \quad (23)$$

where I_m denotes the $(m \times m)$-dimensional identity matrix. The algorithm is obtained by using the matrix inversion lemma to recursively invert the matrix $\sum_{l=1}^k \lambda^{k-l}\Delta q_l \Delta q_l^\top$; see [13] for the derivation.

Remark 3: Note that in order to initialize the algorithm in (22), we need to compute $\partial h(\varphi, w)/\partial\varphi$ for $\varphi = q_0$ and $w = w_0^\circ$. Unlike the computation of S_{k-1} in (14), in this case, we can indeed compute $\widehat{S}_0$ by manipulating the power flow Jacobian without necessarily solving the power flow equations because q_0 and w_0° are known. Furthermore, the initialization of $\widehat{S}_0$ as in (23) is not crucial as we will see in the numerical simulation results presented in Section VII.

Remark 4: Note that if the algorithm in (22) were to be executed for $k > m$, with $\widehat{S}_m = \widehat{S}_m^*$ obtained using (21) for $k = m$, and

$$F_m = \left(\sum_{l=1}^{m} \lambda^{k-l}\Delta q_l \Delta q_l^\top\right)^{-1},$$

the sequence generated by the algorithm would be $\left\{\widehat{S}_l^*\right\}_{l\geq m}$, i.e., for each $k > m$, the algorithm would generate the exact solution to (20) as given in (21).

V. Online Feedback Optimization

Now, on the one hand, by replacing S_{k-1} by $\widehat{S}_{k-1}$ in (13), we obtain that

$$q_k = \Big[q_{k-1} + \gamma_k \widehat{S}_{k-1}^\top\big(v^* - v_{k-1}\big)\Big]_{\underline{q}_k}^{\overline{q}_k}, \quad k \geq 1. \quad (24)$$

On the other hand, by inspecting (21), one can see that an estimate of $\widehat{S}_k$ can be computed if the matrix

$$\sum_{l=1}^{k} \lambda^{k-l}\Delta q_l \Delta q_l^\top$$

is invertible; this can be ensured as mentioned earlier if the sequence $\left\{\Delta q_l\right\}_{l=1}^k$ is persistently exciting. However, it is not clear a priori that (24) will generate such persistently exciting sequence. To address this issue, we modify (24) to add a mechanism to ensure that indeed the sequence $\left\{\Delta q_l\right\}_{l=1}^k$ is persistently exciting. First, by assuming that $q_{k-1} \in [\underline{q}_{k-1}, \overline{q}_{k-1}]$, we can rewrite (24) as follows:

$$q_k = q_{k-1} + \Big[\gamma_k \widehat{S}_{k-1}^\top\big(v^* - v_{k-1}\big)\Big]_{\underline{\Delta q}_k}^{\overline{\Delta q}_k}, \quad k \geq 1, \quad (25)$$

where $\underline{\Delta q}_k = \underline{q}_k - q_{k-1}$ and $\overline{\Delta q}_k = \overline{q}_k - q_{k-1}$. Then, following the ideas in [4], we will make three modifications to (25) as follows:

M1. The term inside the projection term on the right hand side of (25) is modified as follows:

$$\gamma_k \widehat{S}_{k-1}^\top\big(\Delta v_k^* + c z_k\big), \quad (26)$$

where c is some constant, $\Delta v_k^* = v^* - v_{k-1}$, and

$$z_k = \begin{cases} 0, \text{ if } \{\Delta v_l^*\}_{l=1}^k \text{ is persistently exciting,} \\ \text{sampled from } (-\Delta v_k^*)U(0, a_1), \text{ otherwise,} \end{cases} \quad (27)$$

with $a_1 \in (0,1)$ and $U(x,y)$ denoting the continuous uniform distribution over the interval (x,y).

M2. The incremental lower and upper capacity limit in (25) are respectively set to

$$\begin{aligned} &\underline{q}_k + \underline{\eta}_k, \\ &\overline{q}_k - \overline{\eta}_k, \end{aligned} \quad (28)$$

where $\underline{\eta}_k$ is sampled from $U(0, a_2|\underline{\Delta q}_k|)$ and $\overline{\eta}_k$ is sampled from $U(0, a_2|\overline{\Delta q}_k|)$, with $a_2 \in (0,1)$.

M3. By using the two modifications above, define the following quantity:

$$\widetilde{\Delta q}_k = \Big[\gamma_k \widehat{S}_{k-1}^\top\big(\Delta v_k^* + c z_k\big)\Big]_{\underline{\Delta q}_k + \underline{\eta}_k}^{\overline{\Delta q}_k - \overline{\eta}_k}. \quad (29)$$

Let $\mathrm{Null}\big(\widehat{S}_k\big)$ denote the null space of the matrix $\widehat{S}_k$. Then, if the sequence $\left\{\left\{\Delta q_l\right\}_{l=1}^{k-1}, \widetilde{\Delta q}_k\right\}$ is persistently exciting, we update the value of q_k as follows:

$$q_k = q_{k-1} + \widetilde{\Delta q}_k, \quad k \geq 1, \quad (30)$$

otherwise we update its value as follows:

$$q_k = q_{k-1} + \widetilde{\Delta q}_k + \gamma_k \alpha_k \nu_k, \quad k \geq 1, \quad (31)$$

with α_k sampled from $U(-b_k, b_k)$, where $b_k \geq 0$ is arbitrarily chosen so that

$$\underline{\Delta q}_k \leq \widetilde{\Delta q}_k + \gamma_k b_k \nu_k \leq \overline{\Delta q}_k, \quad (32)$$

$$\underline{\Delta q}_k \leq \widetilde{\Delta q}_k - \gamma_k \alpha_k \nu_k \leq \overline{\Delta q}_k, \quad (33)$$

for some arbitrarily chosen $\nu_k \in \text{Null}(\widehat{S}_k)$; this ensures that $\underline{q}_k \leq q_k \leq \overline{q}_k$.

The rationale behind the modifications above is to introduce excitation across the entire space in which the Δq_k's take values. To see this, first consider the case when the capacity constraints are not active at instant k, i.e., $\widetilde{\Delta q}_k = \gamma_k \widehat{S}_{k-1}^\top (\Delta v_k^* + c z_k)$. Let $\text{Row}(\widehat{S}_k)$ denote the row space of $\widehat{S}_k \in \mathbb{R}^{n \times m}$. Then, the idea behind adding the term $c z_k$ to Δv_k^* in Modification M1 is to introduce excitation in the subspace spanned by the columns of $\widehat{S}_k^\top$, i.e., $\text{Row}(\widehat{S}_k)$, whereas the idea behind adding the term $\gamma_k b_k \nu_k$ in Modification M3 is to introduce excitation in $\text{Null}(\widehat{S}_k)$, which is the orthogonal complement of $\text{Row}(\widehat{S}_k)$. Then, since for any $x \in \mathbb{R}^m$ we have that $x = u + v$, with $u \in \text{Row}(\widehat{S}_k)$ and $v \in \text{Null}(\widehat{S}_k)$, this mechanism allows us to introduce excitation in separate parts of the whole space $\mathbb{R}^m$ as needed. The idea behind modifying the incremental capacity limits as described in Modification M2 is as follows. If the capacity constraints are active at instant k, by modifying the incremental capacity limits as in (28), we are leaving headroom to add the term $\gamma_k b_k \nu_k$ if needed, while still ensuring that the constraints in (32) – (33) imposed by the actual incremental capacity limits are satisfied.

VI. Convergence Analysis

Next, we provide the conditions under which the proposed online feedback optimization controller in (29) – (33) converges. To this end, we first need to rewrite the model in (1) as follows. Define $\delta v_k = v_k - v^*$, $\delta q_k = q_k - q_0$, and $\delta w_k = w_k - w_0^\circ$ (note that $w_k - w_0^\circ = w_k^\circ - w_0^\circ + \xi_k$); then, we have that

$$v^* + \delta v_k = h(q_0 + \delta q_k, w_0^\circ + \delta w_k), \quad k \geq 0. \tag{34}$$

Now by using the Taylor series expansion to expand the right hand side of (34) about (q_0, w_0°), we obtain that

$$v_k = S_\varphi q_k + S_w \xi_k + \mu_k + \eta_k, \tag{35}$$

where

$$\begin{aligned} S_\varphi &= \left.\frac{\partial h(\varphi, w)}{\partial \varphi}\right|_{\varphi = q_0,\ w = w_0^\circ} \\ S_w &= \left.\frac{\partial h(\varphi, w)}{\partial w}\right|_{\varphi = q_0,\ w = w_0^\circ} \\ \mu_k &= v^* - S_\varphi q_0 + S_w (w_k^\circ - w_0^\circ), \end{aligned} \tag{36}$$

with η_k representing higher-order terms in δq_k and δw_k.

The next result, the proof of which can be found in [13], shows that the sequence $\{q_k\}_{k \geq 1}$ generated by (27) – (33) converges almost surely to a solution of the optimization problem in (7) provided that the sensitivity estimates, $\{\widehat{S}_k\}_{k \geq 1}$, are unbiased. In doing so, we set $c = 0$ in (29). Also, we assume that (i) $\mu_k = \mu,\ k \geq 0$, where μ is some constant, which is consistent with the assumption we made earlier that w_k° slowly changes with k; and (ii) $\eta_k = 0,\ k \geq 0$, which is reasonable since this term captures the higher-order terms of the Taylor series expansion of $h(q, w)$ about (q_0, w_0°). We plan to address these issues in future work.

Proposition 1: Consider the online feedback optimization controller in (27) – (33) with $c = 0$, and where the sequence $\{\widehat{S}_k\}_{k \geq 0}$ are the unbiased estimates of the sensitivity matrix S_φ, namely, $\text{E}\left[\widehat{S}_k - S_\varphi \,|\, \mathcal{F}_k\right] = 0$, with $\mathcal{F}_k$ denoting the accumulated collection of states up to instant k, $\{(v_l, q_l)\}_{l=0}^k$, and $\text{E}\left[\left\|\widehat{S}_k - S_\varphi\right\|^2\right] \leq \sigma_\varphi$, with σ_φ denoting some positive constant. Let $\mathcal{X}_k^*$ denote the set of optimal solutions of (7). We assume that the following conditions hold:

(a) $\underline{q}_k$ and $\overline{q}_k$ are constant, μ_k is constant, $\text{E}[\eta_k] = 0$ and $\text{E}[\|\eta_k\|_2^2] \leq \sigma$.

(b) γ_k is a diminishing step size, namely,

$$\sum_{k=1}^{\infty} \gamma_k = \infty, \quad \sum_{k=1}^{\infty} \gamma_k^2 < \infty. \tag{37}$$

Then, the sequence $\{q_k\}_{k \geq 1}$ converges almost surely to some point in $\mathcal{X}_k^*$.

The next result, the proof of which can be found in [13], provides the cost error bound at the time instant when the sequence $\{\Delta q_k\}_{k \geq 1}$ is no longer persistently exciting. This indeed will be the case if the rLSE in (22) is used to generate the sequence $\{\widehat{S}_k\}_{k \geq 1}$ as $\widehat{S}_k$ will cease to be an unbiased estimate of the sensitivity matrix, S_φ, as $k \to \infty$.

Proposition 2: Suppose we have for some α and $T > 0$ that

$$\sum_{l=1}^{T} \lambda^{T-l} \Delta q_l \Delta q_l^\top < \alpha I. \tag{38}$$

Suppose that $q_T > \underline{q}_T + \underline{\eta}_T$ and $q_T < \overline{q}_T - \overline{\eta}_T$. Then, the following relation holds for any $q^* \in \mathcal{X}_T^*$:

$$f(q_T) - f(q^*) < (\overline{q}_T - \underline{q}_T) \frac{\sqrt{m\alpha} + B}{\gamma_T}, \tag{39}$$

where B is an upper bound for $\|\zeta_l\|$, $l \geq 1$.

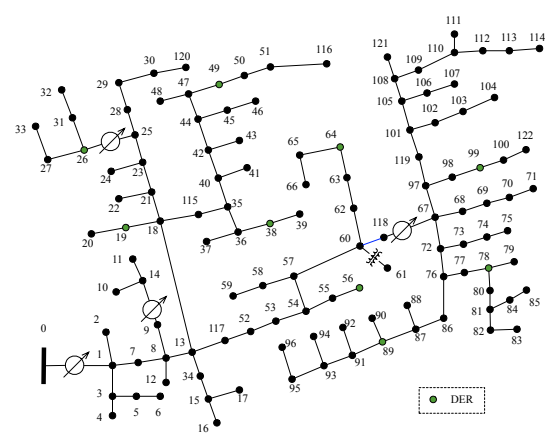

Fig. 1: IEEE 123-bus distribution test feeder.

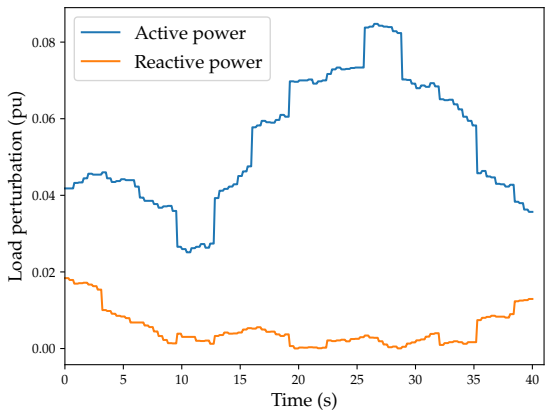

Fig. 2: Trajectories of active and reactive power at bus 1.

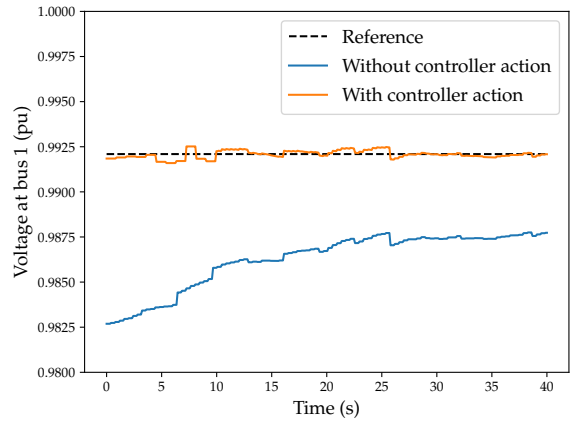

Fig. 3: Trajectory of voltage magnitude at bus 1 for c = 0.5.

VII. Numerical Results

Here, we present numerical simulation results illustrating the effectiveness of our proposed online feedback optimization controller for voltage regulation in power distribution systems. To this end, we employ the test system depicted in Fig. 1, which is a modified version of the three-phase balanced IEEE 123-bus distribution test feeder presented in [3], [14], with reactive-power-capable DERs added to the network at the following buses: 19, 26, 38, 49, 56, 64, 78, 89, and 99. At other buses in the distribution network, random perturbations in the active and reactive power demand are introduced every 100 milliseconds (see Fig. 2 for an illustration of the active and reactive power demand at bus 1).

In this case we have that $n = 122$ and $m = 9$. Thus, we set $F_0 = I_9$ and $\widehat{S}_0$ to the matrix that results from removing the last 113 columns of the matrix I_{122}. Additionally, all the components of $\underline{q}_k$ and $\overline{q}_k$ are set to -0.5 pu and 0.5 pu, respectively, for all k. We also set $a_1 = 0.8$, $a_2 = 0.8$, $\lambda = 0.995$ and $\gamma_k = 0.95$, $k \in \{1, \ldots, 100\}$, $\gamma_k = 0.1$, $k \in \{101, \ldots, 400\}$. Figure 3 show the trajectory followed by bus 1 voltage magnitude under i) no control action, i.e., $q_k = q_0$ for all $k \geq 1$, and ii) the action of the proposed controller for $c = 0.5$. All our simulations demonstrate that the proposed controller is effective at regulating bus voltage magnitudes (see [13] for additional numerical results).

VIII. Concluding Remarks

In this paper we have proposed a controller for voltage regulation in power distribution networks using reactive-power capable DERs. The proposed controller is based on a PSGD algorithm for solving online a sequence of optimization problems, each of which capturing the objectives and constraints of the voltage regulation problem at a particular time instant. In order to execute the controller, it is necessary to know the sensitivities of changes in bus voltage magnitudes with respect to changes in reactive power injections; we assume there are not a priori known and use a rLSE to estimate them. Under certain simplifying assumptions, we showed that the the sequence of DER setpoints generated by the controller converges almost surely to a solution of said optimization problem when the estimates used by the controller are unbiased. In this regard, while the estimates generated by the rLSE will initially be unbiased, in the limit this will no longer be the case; however, the simulation results show that the controller still performs satisfactorily.

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